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Ground states and spectral properties in quantum field theories

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Abstract

In this thesis we consider non-relativistic quantum electrodynamics in dipole approximation and study low-energy phenomenons of quantum mechanical systems. We investigate the analytic dependence of the lowest-energy eigenvalue and eigenvector on spectral parameters of the system. In particular we study situations where the ground-state eigenvalue is assumed to be degenerate. In the first situation the eigenspace of a degenerate ground-state eigenvalue is assumed to split up in a specific way in second order formal perturbation theory. We show, using a mild infrared assumption, that the emerging unique ground state and the corresponding ground-state eigenvalue are analytic functions of the coupling constant in a cone with apex at the origin. Secondly we analyse the situation that the degeneracy is protected by a set of symmetries for the considered quantum mechanical system. We prove, in accordance with known results for the non-degenerate situation, that the ground-state eigenvalue and eigenvectors depend analytically on the coupling constant. In order to show these results we extend operator-theoretic renormalization to such degenerate situations. To complement the analyticity results we additionally show that an asymptotic expansion of the ground state and the ground-state eigenvalue exists up to arbitrary order. The infrared assumption needed for the asymptotic expansion is weaker than the usual assumptions required for other methods such as operator theoretic renormalization to be applicable.

Zusammenfassung in deutscher Sprache

In der vorliegenden Arbeit betrachten wir Modelle der nicht-relativistischen Quantenelektrodynamik in Dipol-Approximation und studieren Phänomene, die bei kleinen Energien in diesen quanten-mechanischen Systemen auftreten. Wir untersuchen die analytische Abhängigkeit des kleinsten Energie-Eigenwertes und der zugehörigen Eigenvektoren im Bezug auf spektrale Parameter des Systems. Insbesondere untersuchen wir Situationen, in denen der Grundzustandseigenwert entartet ist. In der ersten Situation wird angenommen, dass sich der Eigenraum des Grundzustandseigenwertes in zweiter Ordnung in formaler Störungstheorie in bestimmter Weise aufspaltet. Wir zeigen unter Annahme einer schwachen Infrarot-Bedingung, dass der durch die Aufspaltung entstandene nicht-entartete Grundzustand und die zugehörige Grundzustandsenergie analytische Funktionen der Kopplungskonstante in einem Kegel mit Spitze im Ursprung sind. Im zweiten Fall untersuchen wir die Situation, dass die Entartung durch eine Menge von Symmetrien im Systems erzeugt wird. Unter bestimmten Voraussetzungen zeigen wir, dass der Grundzustand und die Grundzustandsenergie analytische Funktionen der Kopplungskonstante sind. Dieses Ergebnis stimmt mit entsprechenden Ergebnissen für nicht-entartete Situationen überein. Um diese Resultate zu zeigen erweitern wir die Methode 'operator-theoretic renormalization' auf diese entarteten Situationen. Ergänzend zu den obrigen Analytizitätsergebnisse zeigen wir, dass eine asymptotische Entwicklung, zu beliebiger Ordnung, des Grundzustandes und der Grundzustandsenergie existiert. Die dafür benötigte Infrarot-Bedingung ist schwächer als die üblichen Bedingungen die gebraucht werden damit andere Methoden, wie zum Beispiel 'operator-theoretic renormalization', anwendbar sind.

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1 Introduction

The general theory of quantum mechanics, which includes the quantum theory of fields is a fundamental theory of nature. This theory gives us a way to describe what happens at small scales and energy levels of atoms and subatomic particles. For example the process of absorption and emission of photons by electrons that are bounded to an atomic nucleus can be described using tools from quantum mechanics. Physicists already worked a lot on important and interesting questions involving quantum mechanics. Moreover a part of the physics community has shifted its research focus to questions concerning relativistic field theories and string theory. From a mathematical point of view even the seemingly simple question: "Does the lowest-energy eigenvalue of a system of non-relativistic matter that interacts with a quantized field of massless particles depend analytically on the spectral parameters of the overall system?" is conceptually very complicated and there are still open problems concerning this question. Likewise similar questions regarding excited energy eigenvalues of such interacting matter-radiation systems do only have partial answers. In this thesis we extend pre-existing answers concerning the above mentioned question to degenerate situations. In particular, we generalize the operator-theoretic renormalization group method so that we can handle Hamiltonians with degenerate eigenvalues. Such renormalization group methods play an essential role in the spectral analysis of quantum mechanical systems.

About 100 years ago Erwin Schrödinger, Werner Heisenberg, Max Born and others developed an early version of quantum theory that is based on work of Max Planck on black body radiation [113] and Albert Einstein on the photoelectric effect [44]. Later Paul Dirac, David Hilbert, John von Neumann and Hermann Weyl developed a mathematically more rigorous formulation of quantum mechanics [43, 130, 132]. A quantum theory of electrodynamics was subsequently formulated. This theory is one of the best studied physical theories of all time. A major break through of quantum electrodynamics was the possibility to describe in a 'robust' way the annihilation and creation process of particles. But still some difficulties remained. In particular in higher order perturbation theory infinities emerged that rendered the computations meaningless. The ultraviolet divergence and the infrared divergence are examples, where the latter has its origin in the fact that photons are massless and the former is related to the Rayleigh-Jeans catastrophe of classical mechanics. This divergent terms led to discrepancies between theoretical predictions and experimental data that could not be explained. A famous example is the derivations in the Lamb shift of energy levels of hydrogen [97]. A solution for this problem was given by Hans Bethe [27]. His idea was to attach to the respective perturbative corrections of mass and charge the emerging infinities. Since mass and charge have finite values by experiment, the infinities get absorbed or rather cut-off by these constants and one derives finite result that are in good agreement with experiments. This procedure is known as renormalization.

Since then many phenomenons of quantum mechanical systems like the Stark effect, the Zeeman effect and scattering effects like Compton and Rayleigh scattering were extensively investigated using the renormalized version of quantum electrodynamics [35, 118, 131]. And a remarkable precision between theoretical predictions of quantum electrodynamics and experiments was achieved. The development of related quantum field theories resulted in sometimes even more sophisticated formulations of quantum mechanics. For example, the path integral formulation, the C*-algebra formalism, quantum field theories in curved space time and statistical models of quantum mechanics [63, 119]. In all of these formulations there exist some kind of renormalization. Hence, in order to systematically handle the renormalization procedure in such quantum field theories a mathematical tool called renormalization group was developed [28, 134]. It was successfully used to establish the so-called standard model of particle physics, a gauge quantum field theory containing the internal symmetries of the unitary product group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, cf. [112].

Over the time different renormalization group methods were developed and many of them use a renormalization group transformations that acts on objects like propagators, partition functions or correlation functions. Moreover non-relativistic approximations of quantum electrodynamics were introduced. Especially the standard model of non-relativistic quantum electrodynamics is an acceptable approximation of quantum electrodynamics in the low energy regime [128], in particular if the energy is below the electron-positron pair production threshold. In connection with this non-relativistic description Bach, Fröhlich and Sigal [20] developed an operator-theoretic renormalization group method where the main new feature was that the renormalization group transformation acts directly on a space of operators. They used their method to establish mathematically rigorous results concerning the absorption and emission of electromagnetic radiation in systems of non-relativistic quantum mechanical matter [19]. This and related work on non-relativistic matter interacting with a quantized massless radiation field led to a tremendous amount of investigations in the beginning of the 21st century. The existence of ground states, of resonances and other properties of quantum mechanical systems such as dispersion relations, the limiting absorption principle and asymptotic completeness were rigorously studied, see for example [14, 21, 54, 56, 59, 66, 67, 70, 123].

The elaborations achieved so far often treat situations where the eigenvalues are non-degenerate. This is especially the case for investigations on the analytic dependence of eigenstates and eigenvalues on spectral parameters using operator-theoretic renormalization. However, physicists observe that many interesting cases involve degenerate eigenvalues. In such a degenerate situation it is natural to assume that either the eigenspace of the eigenvalue split up in higher order perturbation theory, the famous Lamb-shift, or the eigenspace is preserved since the degeneracy of the eigenvalue is protected by a symmetry of the considered quantum mechanical system.

Based on this observation it seems necessary to give answers to questions like the one stated in the beginning in these degenerate situations, too. In order to do so, we generalize the operator-theoretic renormalization group method and use it to analyse the spectrum of Hamiltonians describing degenerate situations. Specifically we show two quite different analyticity results involving degenerate lowest-energy eigenvalues. In the first case, we assume that an existing degeneracy of the lowest-energy eigenvalue splits up into a new unique lowest-energy eigenvalue and higher-energy eigenvalues if an interaction between the matter-like and the photon-like particles is turned on. Secondly we assume that there exists a symmetry in the matter-like part of the system which generates a degeneracy in the eigenvalues of the matter-like particles. Under suitable assumptions, we demonstrate in this case that a ground state and the ground-state eigenvalue depend analytically on the coupling constant. Whereas in first case we prove that the ground state projection and ground-state eigenvalue depend analytically on the coupling constant only in a cone with the apex at the origin. We note that the simultaneously shown existence results for such eigenvalues have already been established in more generality in [67]. In order to complement the results above we further consider a situation where an analytic expansion may not exist and show existence of an asymptotic expansion under very reasonable assumption. Especially we give explicit formulas for the expansion coefficients of the ground state and ground-state energy.

Throughout this thesis we use units in which the speed of light, the electron mass and the Planck constant divided by 2π are equal to 1, namely $c = m_e = \hbar = 1$. In this system of natural units the electron charge is equal to $-\sqrt{\alpha}$, with fine-structure constant $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$. The distance, energy and time is measured in units of $\hbar/m_e c = 3.86 \times 10^{-11}$cm, $mc^2 = 0.511$MeV and $\hbar/m_e c^2 = 1.29 \times 10^{-21}$s.

Moreover this work is written for a reader who is familiar with functional analysis and in particular operator theory. Most mathematical concepts needed in this thesis are to some extend already defined in the multivolume series on functional analysis methods in modern mathematical physics by Reed and Simon [114–117]. Nevertheless we present necessary definitions and basic results of operator theory whenever it is helpful to clarify the procedure and to make this thesis more readable.

We want to mention that the results presented in this thesis are studied in the framework of generalized Spin-Boson models. An example for such a model can be obtained by an dipole approximations of the standard model of quantum electrodynamics, cf. [128]. Moreover note that for specific choices of coupling functions these models are also known as Nelson models. Although the class of generalized Spin-Boson models does not contain the standard model of quantum electrodynamics, such models are nevertheless at the center of a mathematically rigorous description of quantum effects on the length scale of atomic particles. Generalized Spin-Boson models are widely used to study non-relativistic matter interacting with the quantized radiation field or electrons in a solid interacting with a field of phonons [13]. In particular, various variants of the generalized Spin-Boson model are used in condensed matter physics and the realm of quantum chemistry, see for example [62, 111, 135]. Especially in quantum chemistry it is essential that the state of the considered quantum system depends 'nicely' in some sense on the different parameters of the system. For example, the accuracy of the Born-Oppenheimer approximation for a molecule depends on the regularity of the ground-state energy of the considered system of electrons with respect to the positions of their corresponding nuclei [29, 37]. Hence, the results presented in this thesis are not only an answer to interesting mathematical issues concerning analyticity properties of degenerate eigenvalues, but also give additional insight into the scope of an ubiquitous approximation method in quantum chemistry.

Overview of this thesis

In Chapter 2, we introduce the generalized Spin-Boson model. Its main purpose is to describe quantum mechanical processes that involve a linear coupling of a small system to its environment. We begin the chapter with a short summary of its history. Then we take a closer look on the two distinct part of the model, namely the atomic part and the quantized field. In a third step we take into account 'small' interactions between the atomic and the field part. To be more precise we first consider the atomic part which is used to model the matter-like particles of a quantum mechanical system. Later we assume that the atomic particles are able to interact with each other through exchange of relativistic bosonic particles. These relativistic particles form the quantized field which we represent by a symmetric Fock space. We introduce the notion of a Fock space and define operators like annihilation and creation operators. Then we study interacting quantum systems. We deduce abstract spectral properties for interacting particle-field Hamiltonians and conclude the chapter with a variety of examples for generalized Spin-Boson models. It is worth mentioning that one of the main tools to derive spectral informations for such interacting quantum systems is regular perturbation theory [115].

In Chapter 3, we consider a specific example of a quantum mechanical system that is solvable with regular perturbation theory. Unfortunately, in many other interesting cases regular perturbation theory is not applicable. However there exist other methods that are applicable in these cases. One of these methods is the operator-theoretic renormalization group method [18]. As was already mentioned before, this method was developed in order to mathematically rigorously analyse spectral properties of the model of non-relativistic quantum electrodynamics [19]. Fortunately it can be used for many other related models like the Spin-Boson model as well. We provide a description of the method and finish the chapter with important definitions and a review of very useful technical auxiliaries.

In Chapter 4, we examine two specific Spin-Boson models and permit degeneracy. More precisely, we consider two distinct cases. It is reasonable to assume that either a degeneracy of an eigenvalue remains after the interaction is added, or it is lifted at some finite order in formal perturbation theory. In the first case we deal with a degeneracy that is lifted in higher order perturbation theory. This phenomenon is known as the Lamb shift [97]. For simplicity we assume that the degeneracy is lifted at second order in formal perturbation theory once a small interaction is added. In the second case, the degeneracy is induced by a set of symmetries of the atomic system and hence remains if a small interaction is added. In order to keep notation simple we concentrate our analysis in both cases on the lowest-energy eigenvalue and corresponding eigenstates. In the first case we take a look at an uncoupled system consisting of atomic particles and a quantized field of radiation. We suppose that the particle system has originally a degenerate ground-state eigenvalue and assume that this degeneracy is lifted in formal second order perturbation theory when an interaction with the radiation field is introduced. We prove that the coupled system has a ground state and a ground-state eigenvalue that both depend analytically on the coupling constant in an open cone with apex at the origin. For the second case we consider a related problem. Namely, we consider a quantum mechanical system consisting of atomic particles linearly coupled to a bosonic radiation field. This system is subject to specific symmetry restrictions. In particular the system exhibits a degeneracy in the ground-state eigenvalue that is induced by a set of symmetries. We prove that there exists a unique lowest-energy eigenvalue and that this eigenvalue and the corresponding eigenvectors depend analytically on the parameters of the considered system. Moreover, we present a detailed example for such a degenerate system. We note that for these cases, existence of a ground state and ground-state eigenvalue are already known in the literature [60, 67, 99, 127]. However, the analyticity results for such degenerate eigenvalues are new and present a more thorough answer to the question stated in the beginning.

In Chapter 5 we analyse a situations where an analytic expansion may not exist. Hence we study asymptotic expansions in the coupling constant. More precisely we prove, for models of massless quantum fields, under fairly general conditions the existence of an asymptotic expansion for the ground-state eigenvalue. The key idea in the proof is to show that the infinities involved in calculating the Rayleigh-Schrödinger expansion coefficients cancel out. In particular we show that these cancellations can be controlled to arbitrary order without any analyticity assumption. Whereas the existence of an asymptotic expansion is weaker than the existence of an analytic expansion, the presented result holds in situations where analytic expansions have not yet been shown.

2 The Spin-Boson model

In this chapter we introduce the *generalized Spin-Boson model.* It was first mentioned in the work of Arai and Hirokawa [13]. It is a generalization of the standard Spin-Boson model which was widely used to describe the interaction of a localized degree of freedom with a field. A typical example is the interaction of a spin with a bose field [125]. Such a spin is a two-level system and can be described by matrices. The bose field can be realized as infinitely many harmonic oscillators. As was mentioned in [5], the Spin-Boson model is a model describing the coupling of a small system (e.g. a molecule) to its environment. With help of this model classical observables can be derived in a true quantum system. It is worth mentioning that this model is only an approximation. We give an example for this in Chapter 4.
We already mentioned that the standard Spin-Boson model consists of two distinct parts, one with a finite degree of freedom and the other with infinitely many degrees of freedom, which are connected through some model-dependent interaction. The generalized Spin-Boson model has the same structure but the so-called *atomic space* may have more than just one localized degree of freedom. The precise definition of this space depends on the quantum mechanical system that one wants to model. The second part is the so-called *symmetric Fock space.* Detailed definitions of these spaces are given in the subsequent sections.

2.1 The atomic space

The name *atomic space* is retrieve from physics. Specifically using the theory developed by John von Neumann [130] we can identify every point φ in the Hilbert space $\mathcal{H}_{\mathrm{at}}$ with a possible state of an atomic system that is modeled by this Hilbert space. Note that $\varphi \in \mathcal{H}_{\mathrm{at}}$ is often called *wave function.* For example we could choose the space $\mathcal{H}_{\mathrm{at}}^{\mathrm{Fermi}} = \otimes_{\mathrm{anti}}^{N} L^2(\mathbb{R}^3, dx; \mathbb{C}^2) \cong \otimes_{\mathrm{anti}}^{N} \left(L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2\right)$ to model N electrons with spin (*Fermi statistics*). Note that $\otimes_{\mathrm{anti}}^{N}$ denotes the N-fold antisymmetry tensor product, $\mathbb{R}^3$ encodes the spacial coordinate, dx is the Lebesgue measure, and $\mathbb{C}^2$ accommodates the spin of the electron. In general the atomic space $\mathcal{H}_{\mathrm{at}}$ should at least be an arbitrary separable Hilbert space. Since in that case there exists an inner product on $\mathcal{H}_{\mathrm{at}}$, which allow us to measure distances and angles between wave functions, and we can always find a countable orthonormal basis for the Hilbert space $\mathcal{H}_{\mathrm{at}}$, cf. [117, Theorem II.7].
We can not directly measure the actual state of a quantum mechanical system. But there exist directly measurable quantities in quantum mechanical systems, so-called *observables.* The process of measuring one of these observable results in 'breaking down' the involved wave functions to eigenfunctions of a self-adjoint, densely defined operator that is associated to the measured observable [35].

Operators on Hilbert spaces

In this small excursion we state some basic properties of linear operators on Hilbert spaces.

Definition 2.1.1. A *bounded linear / antilinear operator* T between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is a linear / antilinear transformation from $\mathcal{H}_1$ to $\mathcal{H}_2$ for which there exists a constant $M > 0$ such that for all $\psi \in \mathcal{H}_1$,

$$\|T\psi\|_{\mathcal{H}_2} \leq M \|\psi\|_{\mathcal{H}_1} .$$

As usual we denote the space of bounded linear operators by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ where the norm on this space is given by

$$\|T\| = \sup_{\psi \in \mathcal{H}_1, \psi \neq 0} \frac{\|T\psi\|_{\mathcal{H}_2}}{\|\psi\|_{\mathcal{H}_1}} .$$

In the case $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$ we use the shorthand $\mathcal{L}(\mathcal{H})$ to denote the bounded linear operators on $\mathcal{H}$.

Remark 2.1.2. Many operators considered in this thesis are unbounded operators.

Definition 2.1.3. A (possibly unbounded) *linear operator* $T : D(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ is a linear map from its *domain* $D(T)$, a linear subspace of $\mathcal{H}_1$, to its *range* $\mathrm{Ran}(T) \subseteq \mathcal{H}_2$. Moreover an antilinear operator $\tilde{T} : D(\tilde{T}) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ is an antilinear map from its domain to its range.

Definition 2.1.4. We denote by $T \restriction A$ the restriction of a linear operator T to a subset $A \subseteq D(T)$ of its domain. Deviating from this definition we denote by $\mathbb{1}_A$ the identity operator on a vector space A.

Remark 2.1.5. We omit the subscript $_A$ whenever it is clear from the context on which (sub-)space the operator $\mathbb{1}_A$ is defined. For notational simplicity we omit the remaining $\mathbb{1}$ in many cases as well.

Definition 2.1.6. For linear operators T, U we denote by

$$[T, U] := TU - UT\,,$$

the *commutator* of T and U.

Definition 2.1.7. Let T and U be densely defined linear operators on a Hilbert space $\mathcal{H}$. Suppose that:

(i) $D(T) \subset D(U)$,

(ii) For some a and b in $\mathbb{R}$ and all $\varphi \in D(T)$,

$$\|U\varphi\| \leq a\|T\varphi\| + b\|\varphi\|, \tag{2.1}$$

then U is said to be T*-bounded*. The infimum of such a is called the *relative bound* of U with respect to T. If the relative bound is zero, we say that U is *infinitesimally bounded* with respect to T.

Definition 2.1.8. Let T be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Let $D(T^*)$ be the set of $\varphi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ with

$$\langle T\psi, \varphi\rangle = \langle \psi, \eta\rangle \qquad \text{for all} \quad \psi \in D(T)\,.$$

For each such $\varphi \in \mathcal{H}$, we define $T^*\varphi = \eta$ and call T^* the *adjoint* of T. Moreover by Theorem II.4 in [117] (the Riesz lemma) we have that $\varphi \in D(T^*)$ if and only if $|\langle T\psi, \varphi\rangle| \leq C\|\psi\|$ for all $\psi \in D(T)$.

Definition 2.1.9. Let $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a closed operator on a Hilbert space $\mathcal{H}$. A complex number λ is in the *resolvent set* $\rho(T)$, if $\lambda\mathbb{1}_{\mathcal{H}} - T$ is a bijection of $D(T)$ onto $\mathcal{H}$ with a bounded inverse. For $\lambda \in \rho(T)$ we call the bounded operator

$$R_\lambda(T) := (\lambda\mathbb{1}_{\mathcal{H}} - T)^{-1}\,,$$

the *resolvent* of T at λ. If $\lambda \neq \rho(T)$, then λ is said to be in the *spectrum* $\sigma(T)$ of T.

Definition 2.1.10. A densely defined operator T on a Hilbert space $\mathcal{H}$ is called *symmetric* if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$. Equivalently, T is symmetric if and only if

$$\langle T\varphi, \psi\rangle_{\mathcal{H}} = \langle \varphi, T\psi\rangle_{\mathcal{H}} \quad \text{ for all } \varphi, \psi \in D(T).$$

Definition 2.1.11. T is called *self-adjoint* if $T = T^*$, that is, if and only if T is symmetric and

$$D(T) = D(T^*)\,.$$

Remark 2.1.12. It is important to distinguish between symmetric and self-adjoint operators since the spectral theorem does not hold for symmetric operators and only self-adjoint operators may be exponentiated to get a *strongly continuous one-parameter unitary group* [117].

This motivates the following notation regarding self-adjoint extensions of symmetric operators.

Definition 2.1.13. A symmetric operator T is called *essentially self-adjoint* if its closure $\overline{T}$ is self-adjoint. If T is closed, a subset $D \subset D(T)$ is called a *core* for T if $\overline{T \restriction D} = T$.

Remark 2.1.14. If T is essentially self-adjoint, then it has one and only one self-adjoint extension [117].

The atomic Hamiltonian

One of the most important observables of a quantum mechanical system is the total energy. The associated energy operator is called *Hamiltonian.* Let us consider a system of N atomic particles that are given by a wave functions in the Hilbert space $\otimes_{\text{anti}}^N L^2(\mathbb{R}^3)$. The total energy of this system of particles can be determined by acting on the wave function with the *atomic Hamiltonian*

$$H_{\text{at}} := -\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + V(x)\,,$$

where m_j is the mass of the j-th particle, Δ_{x_j} is the Laplacian, x_j is the position of the j-th particle and $V(x)$ is the total potential of the N particles. For a nice enough potential $V(x)$ and a suitable chosen domain $D(H_{\text{at}})$ this operator is essentially self-adjoint. Moreover, the eigenvalues of the closure, provided they exist, are real-valued and therefore 'physically' measurable quantities. This is the ideal and simplest situation. From the mathematical point of view we can use less assumptions and get more general results. This includes the possibility that H_{at} is a closed, densely defined operator on an arbitrary separable Hilbert space $\mathcal{H}_{\text{at}}$ that depends analytically on a parameter.

2.2 The Fock space

In the following we introduce the symmetric Fock space $\mathcal{F}$. Note that it is also known as the bosonic Fock space. It is used to model the field part of the Spin-Boson model, for example a bosonic field of photons or phonons.

2.2.1 Definition of the Fock space

Let $\mathfrak{h}$ be separable Hilbert space over the complex numbers. The elements of this space describe, up to multiplication by a constant, the physical state of a single particle. We use the shorthand $\|\psi\|_{\mathfrak{h}}^2 := \langle\psi,\psi\rangle_{\mathfrak{h}}$. A vector $\widetilde{\psi} = \psi_1 \otimes \cdots \otimes \psi_N \in \otimes^N \mathfrak{h}$ describes a possible state of N particles. Since the N particles are indistinguishable one needs to symmetrize over all possible states to get the actual many-body wave function. For this purpose we denote by $S_n : \otimes^n \mathfrak{h} \to \otimes^n \mathfrak{h}$, $n \geq 1$, the orthogonal projection onto the subspace of $\otimes^n \mathfrak{h}$, which is left invariant by all permutation of the n factors of $\mathfrak{h}$. More explicitly, we choose a basis $\{\phi_k\}$ of $\mathfrak{h}$ and denote by $\mathcal{P}_n$ the permutation group of n elements for $n \geq 1$. Then the operator $\tilde{\pi}$ defined for each $\pi \in \mathcal{P}_n$ is give on the basis elements of $\otimes^n \mathfrak{h}$ as

$$\tilde{\pi}(\{\phi_{k_1} \otimes \phi_{k_2} \otimes \ldots \otimes \phi_{k_n}\}) = \{\phi_{k_{\pi(1)}} \otimes \phi_{k_{\pi(2)}} \otimes \ldots \otimes \phi_{k_{\pi(n)}}\}\,.$$

This extents to a bounded operator on $\otimes^n \mathfrak{h}$ and we can define

$$S_n = \frac{1}{n!}\sum_{\pi \in \mathcal{P}_n} \tilde{\pi}\,. \tag{2.2}$$

Additionally we set $S_0(\otimes^0 \mathfrak{h}) := \mathbb{C}$.

Definition 2.2.1. The *symmetric Fock space* $\mathcal{F}_{\mathfrak{h}}$ is given by

$$\mathcal{F}_{\mathfrak{h}} := \bigoplus_{n=0}^{\infty} S_n(\otimes^n \mathfrak{h})\,.$$

Lemma 2.2.2 (cf. [117])**.** *$\mathcal{F}_{\mathfrak{h}}$ is separable if $\mathfrak{h}$ is separable.*

We omit the subscript $\mathfrak{h}$ and write $\mathcal{F}$ for $\mathcal{F}_{\mathfrak{h}}$ whenever it is clear which space $\mathfrak{h}$ we are considering.
An arbitrary element $\Psi \in \mathcal{F}$ is represented as a sequence

$$\Psi = \left(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots\right) = \{\psi^{(n)}\}_{n=0}^{\infty}\,,$$

with $\psi^{(n)} \in S_n(\otimes^n \mathfrak{h})$. Hence, a vector $\Psi \in \mathcal{F}$ is by construction symmetrized.

Definition 2.2.3. We call $S_n(\otimes^n \mathfrak{h})$ the *n-particle subspace* of $\mathcal{F}$.

Remark 2.2.4. The inner product on $\mathfrak{h}$ extends to an inner product on $\mathcal{F}$, namely

$$\langle \Psi, \Phi \rangle_{\mathcal{F}} := \overline{\psi^{(0)}}\phi^{(0)} + \sum_{n=1}^{\infty} \langle \psi^{(n)}, \phi^{(n)} \rangle_{\otimes^n \mathfrak{h}}, \quad \Psi, \Phi \in \mathcal{F}\,,$$

where the inner product of $\otimes^n \mathfrak{h}$ is given in the usual way (cf. [128]).

Definition 2.2.5. Let $\Psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}$, then Ψ is called *finite particle vector* if $\psi^{(n)} = 0$ for all but finitely many n. The set of all finite particle vectors we denote by $\mathcal{F}_0$.

There is one finite particle vector which actually is a 'no-particle' vector and deserves a special name

Definition 2.2.6. $\Omega := (1, 0, 0, \dots) \in \mathcal{F}_0 \subset \mathcal{F}$ is called the *vacuum vector* in $\mathcal{F}$.

Operators on Fock space

In the following we define special operators on the Fock space $\mathcal{F}$. In particular we define second quantized operators and the so-called annihilation and creation operators.
We begin by defining, for arbitrary $f \in \mathfrak{h}$, the map $b^-(f) : \otimes^n \mathfrak{h} \to \otimes^{n-1}\mathfrak{h}$ by

$$(b^-(f))(\psi_1 \otimes \cdots \otimes \psi_n) = \langle f, \psi_1 \rangle_{\mathfrak{h}} \cdot (\psi_2 \otimes \cdots \otimes \psi_n)\,.$$

This map extends by linearity to finite linear combinations of vectors $\eta = (\psi_1 \otimes \cdots \otimes \psi_n)$. Moreover, for $z \in \mathbb{C}$ we set $b^-(f)(z) = 0$. Thus we extend the map above by $b^-(f) : \mathbb{C} \to 0$.
The following assertion is standard in the literature, cf. [114, Chapter X.7].

Proposition 2.2.7. *Let $\mathfrak{F}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{h}$. The map $b^-(f)$ has the following properties:*

(a) $b^-(f)$ is a bounded operator from $\mathfrak{F}(\mathfrak{h})$ to $\mathfrak{F}(\mathfrak{h})$ with norm $\|f\|_{\mathfrak{h}}$.

(b) $b^+(f) := (b^-(f))^*$ is a map from $\otimes^n \mathfrak{h}$ into $\otimes^{n+1}\mathfrak{h}$ with the action

$$b^+(f)(\psi_1 \otimes \cdots \otimes \psi_n) = f \otimes \psi_1 \otimes \cdots \otimes \psi_n\,.$$

(c) The map $f \mapsto b^-(f)$ is antilinear and $f \mapsto b^+(f)$ is linear.

Proof. For convenience we provide a proof.

(a) Let $\eta \in \otimes^n \mathfrak{h}$, $n \geq 1$, then we have $\|b^-(f)\,\eta\| \leq \|f\|\,\|\eta\|$. Additionally, we get

$$\sup_{\|\eta\|=1} \|b^-(f)\,\eta\| \geq \|b^-(f)\,(f \otimes \tilde{\eta})\| = \|f\|\,, \qquad \text{for suitable } (f \otimes \tilde{\eta}) \in \otimes^n \mathfrak{h}\,.$$

Thus the map $b^-(f)$ extents to a bounded map from $\otimes^n \mathfrak{h} \to \otimes^{n-1}\mathfrak{h}$ with norm $\|f\|$ for all $n \in \mathbb{N}$. The extension of $b^-(f)$ to $\bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{h}$ is well-defined and again a bounded operator with norm $\|f\|$.

(b) Let $\eta \in \otimes^{n-1}\mathfrak{h}$ and $\phi \in \otimes^n \mathfrak{h}$. The following calculation shows the desired statement by linearity

$$\begin{aligned}
\langle\, \eta,\, b^-(f)\,\phi\,\rangle_{\otimes^{n-1}\mathfrak{h}} &= \langle\, \eta, \langle f, \varphi_1 \rangle_{\mathfrak{h}}\, \varphi_2 \otimes \cdots \otimes \varphi_n\,\rangle_{\otimes^{n-1}\mathfrak{h}} \\
&= \langle f, \varphi_1 \rangle_{\mathfrak{h}} \cdot \langle\, \psi_1 \otimes \cdots \otimes \psi_{n-1},\, \varphi_2 \otimes \cdots \otimes \varphi_n\,\rangle_{\otimes^{n-1}\mathfrak{h}} \\
&= \langle\, f \otimes \psi_1 \otimes \cdots \otimes \psi_{n-1},\, \varphi_1 \otimes \cdots \otimes \varphi_n\,\rangle_{\otimes^n \mathfrak{h}} \\
&= \langle\, (b^-(f))^*\,\eta,\, \phi\,\rangle_{\otimes^n \mathfrak{h}}\,.
\end{aligned}$$

(c) This statement follows directly from the definition of an inner product space (cf. [117, Chapter II.1]) and the linearity of the tensor product. □

Now we define *second quantized operators*. For this let A be a self-adjoint operator on $\mathfrak{h}$ with core D. Moreover, define the set $D_A := \{\Psi \in \mathcal{F}_0 : \psi^{(n)} \in \bigotimes_{k=1}^n D \text{ for each } n\}$. On $D_A \cap S_n(\otimes^n \mathfrak{h})$ we define the operator $d\Gamma(A)$ as the linear combination

$$A \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(n-1)\text{-times}} + \mathbb{1} \otimes A \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(n-2)\text{-times}} + \cdots + \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(n-1)\text{-times}} \otimes A\,.$$

This operator is essentially self-adjoint on D_A and is called *second quantization* of A, cf. [114, Section X.7].
For unitary operators U we define $\Gamma(U)$ as the unitary operator on $\mathcal{F}$ which equals

$$\underbrace{U \otimes \cdots \otimes U}_{n\text{-times}}$$

when restricted to $S_n(\otimes^n \mathfrak{h})$ for $n > 0$. Note that $\Gamma(U)$ equals the identity for $n = 0$.

Remark 2.2.8. If e^{itA} is a continuous unitary group on $\mathfrak{h}$, then $\Gamma(e^{itA})$ is the group generated by $d\Gamma(A)$, i.e. $\Gamma(e^{itA}) = e^{itd\Gamma(A)}$. For additional informations we refer to [117, page 309].

Two very important examples of such second quantized operators are the so-called *number operator* and the *operator of energy*. The number operator is the second quantization of the identity operator $\mathbb{1}_{\mathfrak{h}}$. It acts on a non-zero n-particle wave function $\psi^{(n)} \in S_n(\otimes^n \mathfrak{h})$ by multiplication with the number $n \in \mathbb{N}$,

$$d\Gamma(\mathbb{1}_{\mathfrak{h}})\psi^{(n)} = n\psi^{(n)} .$$

Hence the name *number operator*. A definition of the operator of energy is given in Eq. (2.6) below. Next we define annihilation and creation operators. These operators are very useful since they create and annihilate particles in Fock space. Let $f \in \mathfrak{h}$.

Definition 2.2.9. The *annihilation operator* $a(f)$ on $\mathcal{F}$ with domain $\mathcal{F}_0$ is defined by

$$a(f) := \sqrt{d\Gamma(\mathbb{1}) + 1}\, b^-(f) . \tag{2.3}$$

The *creation operator* $a^*(f) \restriction \mathcal{F}_0$ on $\mathcal{F}_0$ is given by

$$a^*(f) \restriction \mathcal{F}_0 := \bigoplus_{n=0}^{\infty} S_n b^+(f) \sqrt{d\Gamma(\mathbb{1}) + 1} .$$

One can easily verify that $[a(f)]^* = a^*(f)$ on $\mathcal{F}_0$.

Corollary 2.2.10. *The maps $a(f)$ and $a^*(f) \restriction \mathcal{F}_0$ are closable.*

Proof. $a(f)$ is closable since its adjoint is densely defined on $\mathcal{F}_0$ and $D(a(f))$ is dense in $\mathcal{F}$. Similarly we get that $a^*(f)$ is closable on $\mathcal{F}_0$, cf. [114, Theorem VIII.1]. This concludes the proof. □

Remark 2.2.11. We denote the closure of $a(f)$ and $a^*(f)$ again by $a(f)$ resp. $a^*(f)$.

Lemma 2.2.12. *The following relations are valid for arbitrary $f,\, g \in \mathfrak{h}$.*

$$\big[a(f),\, a(g)\big] = \big[a^*(f),\, a^*(g)\big] = 0 , \qquad \big[a(f),\, a^*(g)\big] = \langle f,\, g\rangle . \tag{2.4}$$

Moreover, for all $f \in \mathfrak{h}$ we have

$$a(f)\Omega = 0 .$$

Proof. Since $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}_0$ it follows directly from Definition 2.2.9 that $a(f)\Omega = 0$ for all $f \in \mathfrak{h}$. Now let $\Psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_0$. Using Definition 2.1.6 and 2.2.9 we directly get that

$$\big(a(f)\, a(g) - a(g)\, a(f)\big)\Psi = 0 , \qquad \big(a^*(f)\, a^*(g) - a^*(g)\, a^*(f)\big)\Psi = 0 .$$

Hence it remains to show that $a(f)\, a^*(g)\Psi - a^*(g)a(f)\Psi = \langle f,\, g\rangle\Psi$ in order to verify Eq. (2.4). For this let $\psi^{(n)} \in S_n(\otimes^n \mathfrak{h}) \subset \mathcal{F}_0$. Then

$$\begin{aligned}
\big(a(f)\, a^*(g) - {} & a^*(g)a(f)\big)\psi^{(n)} \\
&= a(f)\big(\sqrt{n+1}\, S_n\, b^+(g)(\psi_1 \otimes \ldots \otimes \psi_n)\big) - a^*(g)\big(\sqrt{n}\, b^-(f)(\psi_1 \otimes \ldots \otimes \psi_n)\big) \\
&= (n+1)\, \big(b^-(f)\big(S_n(g \otimes \psi_1 \otimes \ldots \otimes \psi_n)\big)\big) - n\, b^-(f)\big(S_n\, b^+(g)(\psi_2 \otimes \ldots \otimes \psi_n)\big) \\
&= (n+1)\, \big(b^-(f)\big(S_n(g \otimes \psi_1 \otimes \ldots \otimes \psi_n)\big)\big) - n\, b^-(f)\big(S_n(\psi_1 \otimes g \otimes \ldots \otimes \psi_n)\big) \\
&= \langle f,\, g\rangle\, S_n(\psi_1 \otimes \ldots \otimes \psi_n) .
\end{aligned}$$

Since n was arbitrary the assertion follows by linearity from Definition 2.2.1. □

Remark 2.2.13. The relations (2.4) are known as the *canonical commutation relations* (CCR).

This concludes the general definition of the symmetric Fock space and important associated operators. Next we choose a concrete Hilbert space $\mathfrak{h} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. We call the corresponding Fock space $\mathcal{F}$ the *bosonic Fock space* since we use it in the subsequent chapters to model a quantized field of massless bosonic particles.

2.2.2 The bosonic Fock space

Let $\mathfrak{h} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ and let the inner product of measurable functions $\psi, \phi \in \mathfrak{h}$ be given by

$$\langle \psi,\, \phi \rangle_{\mathfrak{h}} := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \overline{\psi(\mathbf{k},\lambda)}\phi(\mathbf{k},\lambda)\, d^3\mathbf{k}\,.$$

Remark 2.2.14. The norm induced by the inner product satisfies for $\psi \in \mathfrak{h}$

$$\|\psi\|^2_{\mathfrak{h}} := \langle \psi,\, \psi \rangle_{\mathfrak{h}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |\psi(\mathbf{k},\lambda)|^2 d^3\mathbf{k} < \infty\,.$$

The term $|\psi(\mathbf{k},\lambda)|^2$ is interpreted as the probability density of the particle with wave vector ψ having momentum $\mathbf{k}$ and polarisation λ, i.e. $\|\psi\|^2_{\mathfrak{h}}$ denotes the probability to find a particle in state ψ.

To simplify our notation we define

$$k := (\mathbf{k},\lambda)\,, \qquad \int dk := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d^3\mathbf{k}\,, \qquad |k| := |\mathbf{k}|\,. \tag{2.5}$$

Moreover we omit subscripts on norms and inner products whenever it is clear from the context on which (sub-)space the norms and inner products act.

Note that we use in the following, without further comment, that there exists a unique isomorphism from $\left(L^2(\mathbb{R}^3 \times \mathbb{Z}_2)\right)^{\otimes n}$ to $L^2((\mathbb{R}^3 \times \mathbb{Z}_2)^n)$ for all $n \in \mathbb{N}$. We refer to Section II.4 in [117] for more details.

The bosonic Fock space $\mathcal{F}$ is defined as the the symmetric Fock space $\mathcal{F}_{\mathfrak{h}}$ with $\mathfrak{h} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, cf. Definition 2.2.1. Moreover we define the *operator of energy*, also called *free field Hamiltonian*, H_f as the second quantization of the operator M_ω which acts by multiplication with a real-valued function $\omega : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{R}$, i.e.

$$H_f := d\Gamma(M_\omega)\,. \tag{2.6}$$

We make the usual choice for the one-particle dispersion relation ω.

Definition 2.2.15. The *dispersion relation* $\omega : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{R}^+_0$ is defined by

$$\omega(k) := |k|\,.$$

For $\Psi \in \mathcal{F}_0$ we obtain

$$H_f\Psi = d\Gamma(M_\omega)\Psi = \bigoplus_{n=1}^{\infty} \left(\sum_{i=1}^{n} \omega(k_i) \right) \psi^{(n)}(k_1, \ldots, k_n)\,. \tag{2.7}$$

Thus, an element of the one-particle subspace of $\mathcal{F}_0$ has energy $|k|$ and an element of the n-particle subspace $S_n(\otimes^n \mathfrak{h})$ of $\mathcal{F}_0$ has energy $\sum_{i=1}^n |k_i|$. Note that the vacuum vector Ω has energy zero.

Therefore the free field Hamiltonian H_f depends on the momenta $k_1, \ldots, k_n \in \mathbb{R}^3 \times \mathbb{Z}_2$ of the particles in each n-particle subspaces of the bosonic Fock space $\mathcal{F}$. Hence it is very convenient to have a momentum representation for the annihilation and creation operators as well.

Let $\mathcal{S}(\mathbb{R}^3 \times \mathbb{Z}_2)$ denote the space of rapidly decreasing functions on $\mathbb{R}^3 \times \mathbb{Z}_2$. For more information on this space we refer to [117, Chapter V.3]. The map $f \mapsto a(f)$ is antilinear for $f \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{Z}_2)$. Hence using Eq. (2.3) we can write the action of $a(f)$ on $\Psi \in \mathcal{F}_0$ as follows

$$(a(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1} \int \overline{f(k)} \psi^{(n+1)}(k, k_1, \ldots, k_n)\, dk\,.$$

Therefore we can identify

$$a(f) = \int \overline{f(k)}\, a(k)\, dk\,,$$

where $a(k)$ is understood as an unbounded, operator-valued distribution on $\mathbb{R}^3 \times \mathbb{Z}_2$. The domain of this distribution is the set $D_{\mathcal{S}} = \left\{ \Psi \in \mathcal{F}_0 : \psi^{(n)} \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{Z}_2) \right\}$ and it is acting as

$$(a(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1}\, \psi^{(n+1)}(k, k_1, \ldots, k_n)\,. \tag{2.8}$$

We understand this in the sense of quadratic forms, i.e. for $\Phi,\, \Psi \in D_{\mathcal{S}}$

$$f \mapsto \langle \Phi,\, a(f)\Psi \rangle = \int \overline{f(k)} \langle \Phi, a(k)\Psi \rangle dk \tag{2.9}$$

is a distribution. We call this distribution the *smeared annihilation operator* and denote it by $a(f)$. We would like to do exactly the same for the operator $a^*(f)$, but the adjoint of the operator-valued distribution $a(k)$ is not densely defined on $\mathcal{F}$, cf. [114, Chapter X.7]. It is formally given by

$$(a^*(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(k - k_i)\psi^{(n-1)}(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)\,.$$

Fortunately we are interested in $a^*(f)$ only in the sense of quadratic forms and $a^*(k)$ is well-defined as a quadratic form on $D_{\mathcal{S}} \times D_{\mathcal{S}}$. For $\Psi \in \mathcal{F}_0$ we get

$$(a^*(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(k_i)\psi^{(n-1)}(k_1, \ldots, \widehat{k_i}, \ldots, k_n)\,,$$

where $\widehat{k_i}$ means here that k_i is omitted. Again we identify

$$a^*(f) = \int f(k)\, a^*(k)\, dk\,,$$

and call it the *smeared creation operator*. Moreover we extend the definition of smeared annihilation and creation operators to $f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by using that $\mathcal{S}(\mathbb{R}^3 \times \mathbb{Z}_2)$ is dense in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$.

Remark 2.2.16. The operator-valued distribution $a(k)$ and its adjoint $a^*(k)$ also satisfy *canonical commutation relations*,

$$\left[a^\sharp(k), a^\sharp(k')\right] = 0\,, \qquad [a(k), a^*(k')] = \delta(k - k')\,, \tag{2.10}$$

where $a^\sharp = a$ or a^*. For additional informations on the canonical commutation relations and for a more algebraic definition of the (bosonic) Fock space we refer to [41].

A representation of the free field Hamiltonian H_f as a quadratic form on $D_S \times D_S$ in terms of the operator-valued distribution $a(k)$ and its adjoint $a^*(k)$ is formally given by

$$H_f = \int a^*(k)\, \omega(k)\, a(k)\, dk\,. \tag{2.11}$$

Remark 2.2.17. On the Fock space $\mathcal{F}$ the right hand side of Eq. (2.11) defines a densely defined, positive, self-adjoint operator that has a simple eigenvalue 0 corresponding to the vacuum vector Ω and absolutely continuous spectrum on the positive half-axis, cf. [21].

Depending on the free field Hamiltonian H_f we define the following subspace of $\mathcal{F}$

Definition 2.2.18. The *reduced Fock space* is define as the following subspace of the bosonic Fock space

$$\mathcal{H}_{\text{red}} := \text{Ran}\mathbb{1}_{[H_f<1]} = \mathbb{1}_{[0,1]}(H_f)\mathcal{F} =: P_{\text{red}}\mathcal{F}\,. \tag{2.12}$$

Remark 2.2.19. At the end of this section we want to comment on the dispersion relation $\omega(k)$. Let us consider a particle with mass m, which moves freely in one dimension. To describe this particle quantum mechanically we replace the position x and velocity $v = p/m$ of the particle by its probability wave function $\psi(x,t)$ and solve the Schroedinger equation

$$i\hbar\frac{d}{dt}\psi(x,t) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t)\,.$$

A solution of this differential equation is given by $\psi(x,t) = Ae^{i(\omega t - kx)}$ which yields the dispersion relation

$$\omega = \frac{\hbar k^2}{2m}\,.$$

Hence the dispersion relation depends on the solution of the considered differential equation and it is directly connected to a specific property of the quantum mechanical system. Namely whether the system is *dispersionful* or *dispersionless*, i.e. the relation between ω and k is non-linear resp. linear. The *phase velocity* is defined by $v_p := \omega/k$, i.e. the speed of a single traveling wave. The *group velocity*, the speed of a superposition of many traveling waves (sometimes also called a bump) is given by $v_g := d\omega/dk$. In general both velocities are functions of the momentum k and not equal to each other. In the following chapters we use the bosonic Fock space to model a quantized field of massless bosonic particles. Massless particles have a constant velocity c independent of their actual momentum k. Thus the phase velocity is equal to the group velocity up to a constant and ω is linear in k.

2.3 Properties of the Spin-Boson Hamiltonian

In this section we examine spectral properties of the Spin-Boson Hamiltonian. We assume that this system consists of atomic particles in an atomic space and a quantized field of particles modeled by the bosonic Fock space. Before defining the interacting Spin-Boson Hamiltonian (Eq. (2.15)) we first look at the energy spectrum of a non-interacting system. In particular we consider the Hilbert space $\mathcal{H} := \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}$ consisting of a separable Hilbert space $\mathcal{H}_{\mathrm{at}}$ and the bosonic Fock Space $\mathcal{F}$. These spaces were introduced in the last two sections.
Let the atomic Hamiltonian H_{at} be a closed operator on $\mathcal{H}_{\mathrm{at}}$. As in Section 2.2 we denote by H_f the free field Hamiltonian on $\mathcal{F}$ with dispersion relation $w(k) := |k|$. For $\theta \in \{z \in \mathbb{C} | |\operatorname{Im} z| < \pi/4\}$ we define a Hamiltonian on $\mathcal{H}$ by

$$H_0(\theta) := H_{\mathrm{at}} \otimes \mathbb{1} + e^{-\theta} \mathbb{1} \otimes H_f \,. \tag{2.13}$$

In the following we suppose that we know the whole spectral theory of the operator $H_0(\theta)$. For example let us assume for a moment that the following Hypothesis is valid.

(H): The operator H_{at} is self-adjoint and its spectrum consists of a pure point and an absolute continuous part, i.e. $\sigma(H_{\mathrm{at}}) = \sigma_{pp}(H_{\mathrm{at}}) \cup \sigma_{ac}(H_{\mathrm{at}})$.

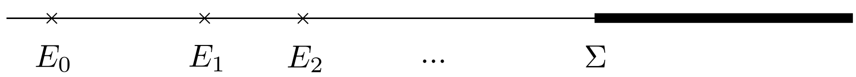


Figure 2.1: The spectrum of H_{at}.

We denote by $\Sigma := \inf \sigma_{\mathrm{ac}}(H_{\mathrm{at}})$ the so-called *ionization threshold* and $E_0 := \inf \sigma_{\mathrm{pp}}(H_{\mathrm{at}})$ is the so-called *ground-state eigenvalue/energy* of the Hamiltonian H_{at}. The spectrum of H_f is also well-known (cf. Remark 2.2.17) and given by $\sigma(H_f) = [0, \infty)$. The factor $e^{-\theta}$ rotates the spectrum of H_f by an angle of $|\operatorname{Im} \theta|$ into the lower half-plane (Fig. 2.2). Note that we technically do not need this factor since we only consider ground-state eigenvalues and related eigenstates, also called *ground states*, in this thesis. In spite of that we keep this factor for a moment and study so-called *resonances*.
A precise mathematical definition of resonances is given by the *Aguilar-Balslev-Combes-Simon theory.* In the following we give a short introduction to this theory. For this let $H = -\Delta + V$ be a linear operator on a Hilbert space $\mathcal{H}$. Such operators are called *Schrödinger operator with potential* V. They are Hamiltonians of suitably chosen quantum systems described by the potential V.

Definition 2.3.1. The *quantum resonances* of a Schrödinger operator H associated with a dense set of vectors $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ are the poles of the meromorphic continuations of all matrix elements $\langle f,\, R_z(H)\, g\rangle$ from $\{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ to $\{z \in \mathbb{C} \,|\, \mathrm{Im} z \leq 0\}$, with $f,\, g \in \mathcal{A}$.

The Aguilar-Balslev-Combes-Simon theory shows that such meromorphic continuations exist and that the poles of this continuation are related to eigenvalues of specific non-self-adjoint operators associated to the Schrödinger operator H. Eigenstates corresponding to these eigenvalues are identified as the resonances states of H. For detailed informations we refer the reader to [4,23,124] and references therein. For a more comprehensive introduction to the spectral theory of Schrödinger operators we refer to [85].
As an example we consider the case of dilation analyticity for the Hamiltonian $H_0(\theta)$ in Eq. (2.13). We introduce a scaling of the position x_j of the atomic particles. More precisely we rescale the result of acting by the operator of position on elements $\psi_{\mathrm{at}} \in \mathcal{H}_{\mathrm{at}}$. Similarly we introduce a scaling of the momenta k of the elements in Fock space $\mathcal{F}$, this means we rescale the energy of all $\Psi \in \mathcal{F}$. These scaling operations are defined as follows

$$x_j \mapsto e^{\theta} x_j \,, \qquad k \mapsto e^{-\theta} k \,.$$

For $\theta \in \mathbb{R}$, this change of scale can be realized as a unitary transformation U_θ on the Hilbert space $\mathcal{H}$. Let $\mathcal{D} \subseteq \mathcal{H}$ be the subspace of all vectors ψ, with the property that $\psi(\theta) := U_\theta \psi$ is analytic in θ. We see then that $U_\theta\, H_f\, U_\theta^* = e^{-\theta} H_f$. Moreover it has been shown in [21, 108] that $\mathcal{D}$ is dense in $\mathcal{H}$ for $|\operatorname{Im} \theta| < \pi/2$. Combining this with well-known results from [4, 23] on dilatation analyticity for Schrödinger operators we deduce that

$$F^0_{\psi,\varphi}(\theta, z) := \langle\, \psi(\overline{\theta})\,,\, (z - H_0(\theta))^{-1} \varphi(\theta)\,\rangle$$

is independent of θ for $|\mathrm{Im}\,\theta| < \pi/2$ and $\psi, \varphi \in \mathcal{D}$. Thus, for fixed $\mathrm{Im}\,\theta$, the spectrum of $H_0(\theta)$ is contained in the shaded region depicted in Figure 2.2. In other words, $F^0_{\psi,\varphi}(\theta, z)$ is analytic in z in the complement of the shaded region. For additional informations on dilatation analyticity we refer to [21, 89].

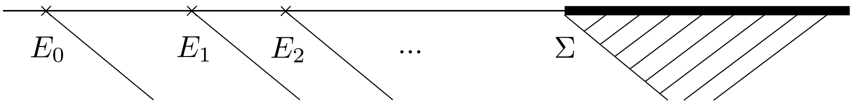


Figure 2.2: Projection of the Riemann surface of $z \mapsto F^0_{\psi,\varphi}(\theta, z)$ onto the energy plane.

The atomic Hamiltonian H_{at} has non-trivial spectrum since it is a closed operator. Hence we deduce, through separation of variables, that the spectrum of $H_0(\theta)$ is given by

$$\sigma(H_0(\theta)) = \overline{\sigma(H_{\mathrm{at}}) + \mathrm{e}^{-\theta}\sigma(H_f)}\,. \tag{2.14}$$

Note that from here on we omit θ whenever the respective estimates are uniform in this parameters. From Hypothesis (H) and Eq. (2.14), we see that the ground-state eigenvalue of H_{at} coinsides with the ground-state eigenvalue of H_0. The big difference between these two eigenvalues is, that the ground-state eigenvalue of H_0 is not isolated from the rest of its spectrum, where as it could have been an isolated eigenvalue of H_{at}. Thus, regular perturbation theory is not applicable. The ground-state eigenvalue lies at the tip of a branch of continuous spectrum of the Hamiltonian H_0. In the case that H_{at} has other pure point eigenvalues in addition to the ground-state eigenvalue, these formerly isolated eigenvalues $E_1, E_2, \ldots$ of H_{at} are imbedded in the continuous spectrum of H_0 where at each E_j is a threshold of a branch of continuous spectrum (Fig. 2.2). A precise mathematical description is given by the above mentioned theory of analytic dilatation.
Regardless of these difficulties we are still interested in understanding the fate of the eigenvalues E_j if an interaction between the atomic space $\mathcal{H}_{\mathrm{at}}$ and the Fock space $\mathcal{F}$ is included. The kind of interactions we consider in this thesis are so-called *small perturbations* W_g of the unperturbed Hamiltonian H_0. Such a small perturbation W_g can be characterized by the following properties.

(i) the interaction is defined on a subset of the domain of the unperturbed Hamiltonian,

$$D(W_g) \subseteq D(H_0)\,.$$

(ii) $W_g(\theta) := U_\theta\, W_g\, U_\theta^{-1}$ is dilatation analytic, hence $\theta \mapsto W_g(\theta)$ is an analytic function on

$$B(0, \vartheta_0) := \{z \in \mathbb{C} \,|\, |z| < \vartheta_0\}\,,$$

for some $\vartheta_0 > 0$.

(iii) W_g is H_0-bounded, specifically for $g > 0$ it obeys the bound

$$\big\|W_g(H_0 + iC)^{-1}\big\| \le g\,\tilde{C}\,,$$

for some constants $C, \tilde{C} \in \mathbb{R}_+$.

These properties ensure that the *Spin-Boson Hamiltonian*

$$H_g := H_0 + W_g \tag{2.15}$$

is self-adjoint and semibounded on $D(H_0)$ for $0 < g < \tilde{C}^{-1}$. Moreover the family of operators

$$\big\{H_g(\theta) : \theta \in B(0, \vartheta_0)\big\}$$

is dilatation analytic. This was proved in [21] for specific choices of H_0 and W_g.
The existence of a unique ground state for the Spin-Boson Hamiltonian H_g was verified in [13] and existence results for resonances are for example given in [19, 20, 91]. More spectral properties of the Spin-Boson Hamiltonian and other related models can be found in [30, 60, 82–84, 88, 125]. We refer also to Chapter 3 for further references.

2.3.1 Examples of generalized Spin-Boson Models

To conclude this section we give examples for models which are in the class of generalized Spin-Boson models. A great variety of physically interesting systems can be modeled although it incorporates only a linear coupling of a small system to its environment.
Let $\mathfrak{h} := L^2(\mathbb{R}^\nu)$ and the Fock space $\mathcal{F}$ defined as in Definition 2.2.1. Moreover, let $\mathcal{H}_{\mathrm{at}}$ be a separable Hilbert space. Note that we choose a more precise atomic space in every example. We define the *Segal field operator* $\Phi_S(f)$ on $\mathcal{F}_0$ by

$$\Phi_S(f) := \frac{1}{\sqrt{2}}\big(a(f) + a^*(f)\big)\,,$$

for $f \in L^2(\mathbb{R}^\nu)$. On $\mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}$ we choose as the unperturbed Hamiltonian

$$H_0 := A \otimes \mathbb{1} + \mathbb{1} \otimes H_f\,,$$

and consider the perturbation

$$W_g := \overline{g\sum_{j=1}^{J} B_j \otimes \Phi_S(f_j)}\,,$$

where the operators A, B_j will be specified in every example separately and $g \in \mathbb{R}$ is a coupling constant. Hence, the Spin-Boson Hamiltonian is represented by

$$H_g = \overline{A \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\sum_{j=1}^{J} B_j \otimes \Phi_S(f_j)}\,.$$

The following examples are from [13].

Example 2.3.2 (The standard Spin-Boson model). Let $\mu > 0$ be a constant and σ_1, σ_3 be the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the case where $\mathcal{H}_{\mathrm{at}} = \mathbb{C}^2$, $A = \mu\,\sigma_3/2$, $J = 1$, $B_1 = \sqrt{2}\,\sigma_1$, $f_1 = f \in L^2(\mathbb{R}^\nu)$. Then H_g takes the form

$$H_g^{\mathrm{SB}} := \frac{\mu}{2}\sigma_3 \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\sqrt{2}\,\sigma_1 \otimes \Phi_S(f)\,,$$

of the standard Spin-Boson model, which acts in $\mathbb{C}^2 \otimes \mathcal{F}$.

Example 2.3.3 (An N-level system coupled to a Bose field). Consider the case where $\mathcal{H}_{\mathrm{at}} = \mathbb{C}^N$ with $N < \infty$ and let $J = 1$. Then we can represent A and $B := B_1$ by $N \times N$ Hermitian matrices, such that A has N eigenvalues, counting multiplicity. Hence A describes an unperturbed 'atom' with N energy levels. Note that a *positive-temperature version* of this model is discussed in [92]. However, the Hamiltonian, in that case, is neither bounded from above nor from below. This makes a big difference.

Example 2.3.4 (A lattice spin system interacting with phonons). Let Λ be a finite set of the ν-dimensional square lattice $\mathbb{Z}^\nu$ and consider the case where an N component spin $S = \big(S^{(1)}, S^{(2)}, \ldots, S^{(N)}\big)$ sits on each site $i \in \Lambda$ and each component $S^{(n)}$ acts on $\mathbb{C}^s$ with $s \in \mathbb{N}$. The Hilbert space of this spin system is given by $\mathcal{H}_\Lambda = \otimes_{i\in\Lambda}\mathcal{H}_i$ with $\mathcal{H}_i = \mathbb{C}^s$, $i \in \Lambda$. The spin at site i is defined by $S_i = \Big(S_i^{(1)}, S_i^{(2)}, \ldots, S_i^{(N)}\Big)$, $S_i^{(n)} = \mathbb{1} \otimes \cdots \otimes S^{(n)} \otimes \mathbb{1} \otimes \cdots \mathbb{1}$ with $S^{(n)}$ acting on $\mathcal{H}_i$. A Hamiltonian of the spin system interacting with a Bose field is given by

$$H_g^\Lambda := \overline{\left(-\sum_{(i,j)\subset\Lambda} J_{ij}\, S_i \cdot S_j\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\sum_{j\in\Lambda}\sum_{n=1}^{N} S_j^{(n)} \otimes \Phi_S(f_j^{(n)})}\,,$$

acting on $\mathcal{H}_\Lambda \otimes \mathcal{F}$, where $J_{ij} \in \mathbb{R}$ are constants, i, $j \in \Lambda$ and $f_j^{(n)} \in L^2(\mathbb{R}^\nu)$, $j \in \Lambda$, $n = 1, \ldots, N$.

Example 2.3.5 (Non-relativistic particles interacting with a Bose Field). A Hamiltonian of N non-relativistic particles with mass $M > 0$ in a potential V (a real-valued measurable function on $\mathbb{R}^{\nu N}$) and in interaction with a Bose field is given by

$$H_g^{PB} = \overline{\left(-\frac{1}{2M}\Delta + V\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\sum_{j=1}^{J} G_j \otimes \Phi_S(f_j)}\,,$$

acting in $L^2(\mathbb{R}^{\nu N}) \otimes \mathcal{F}$, where Δ is the Laplacian on $L^2(\mathbb{R}^{\nu N})$ and G_j, $j = 1, \ldots, J$, are symmetric operators on $L^2(\mathbb{R}^{\nu N})$. Related models were discussed for example in [8–11, 18, 19, 52, 53, 91, 126]. Moreover, additional references for more recent works are given in Chapter 3.

Example 2.3.6 (A model of a Fermi field interacting with a Bose field). Let $\mathcal{F}_a$ be the fermion Fock space over $L^2(\mathbb{R}^\nu; \mathbb{C}^s)$ $(s \geq 1)$, H_a a second quantization operator on $\mathcal{F}_a$ and $\psi(\rho)$, $\rho \in L^2(\mathbb{R}^\nu; \mathbb{C}^s)$, the fermion annihilation operators on $\mathcal{F}_a$ (which are bounded). Then a Hamiltonian of a quantum system of a Fermi field interacting with a Bose field is given by

$$H_g^{FB} = H_a \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \overline{g \sum_{j=1}^{J} \psi(\rho_j)^* \psi(\rho_j) \otimes \Phi_S(f_j)}\,,$$

acting on $\mathcal{F}_a \otimes \mathcal{F}$, where $\rho_j \in L^2(\mathbb{R}^\nu; \mathbb{C}^s)$, $j = 1, \ldots, J$. In case $s = 2$, this model may serve as a model of electrons interacting with phonons in a metal.

In Chapter 4 we consider specific cases of Example 2.3.5.

3 Operator-theoretic renormalization

In the last chapter we discussed various examples of systems that can be described by the generalized Spin-Boson model. Moreover we observed that, in general, regular perturbation theory is no longer applicable. In this chapter, we provide a method to overcome this difficulty. Specifically we elaborate on the *operator-theoretic renormalization group method* proposed in [18] and further developed in [14, 20]. This method was developed for the more general model of non-relativistic quantum electrodynamics, cf. [19]. The generalized Spin-Boson model can be seen as a special case of this model with only linear terms of annihilation and creations operators in the interaction operator.
In Section 3.1 we present an overview on spectral results for such kind of models and provide references for further reading. We start the section with an explicit example of a quantum mechanical system for which regular perturbation theory is applicable. In Section 3.2 we describe the operator-theoretic renormalization group method. In Section 3.3 we introduce Banach spaces of integral kernels and state some useful technical auxiliaries.

3.1 Perturbation theory

Perturbation theory is widely used to calculate various quantities in quantum mechanics. A good approximation of physical properties can be expected if the perturbation is 'small' compared to the unperturbed system. In case of isolated eigenvalues one can apply analytic perturbation theory to calculate the eigenvalues and eigenvectors of the considered quantum mechanical system in terms of convergent power series, which are also known as Rayleigh-Schrödinger perturbation series [93, 115]. However, in models involving massless quantum fields the ground state is in general no longer isolated from the rest of the spectrum and analytic perturbation theory is not applicable.
In this section we first consider a specific quantum mechanical system and apply perturbation theory to derive informations about its spectrum. Then we consider the corresponding general spectral problem and give a brief overview on results that were proven using different methods to cope with the above mentioned problem.

The Stark effect

Detailed explanations and corresponding calculations for the following example can be found in every standard textbook on quantum mechanics, for example [36, 122]. Moreover a rigorous mathematical treatment of the Stark effect for two-body and N-body Hamiltonians is given in [79, 80].

Example 3.1.1. We consider a perturbation of a hydrogen atom by an homogeneous electric field. This effect is known as Stark effect and was discovered in 1913, cf. [129]. Let the Hamiltonian of the unperturbed system be given by

$$H_0 := \frac{p^2}{2m} - \frac{e^2}{r} \,,$$

where p is the momentum operator, m is the mass and e is the charge of the electron. This operator is essentially self-adjoint and semibounded on $D(H_0) = C_0^\infty(\mathbb{R}^3)$, cf. [114]. As a perturbation we choose the following electric field oriented along the z-axis

$$V := \begin{cases} -e\,C\,, & \text{for } z > C\,, \\ -e\,z\,, & \text{for } |z| < C\,, \\ -e\,C\,, & \text{for } z < -C\,, \end{cases}$$

for a constant $C > 0$. We control the strength of the perturbation through a scaling factor $0 < \lambda < 1$. Hence the perturbed Hamiltonian is given by

$$H_\lambda = H_0 + \lambda V ,$$

with $D(H_\lambda) = D(H_0)$. We are interested in the behavior of the eigenvalues of H_0 once the system gets perturbed. We apply analytic perturbation theory and expand the eigenvalues and eigenvectors in power series depending on λ. Hence by equating coefficients we can calculate the corrections to the eigenvalues at every order.
In the following we use Rydberg units of energy [101] and restrict our analysis to the second eigenvalue $E_2 = -\frac{1}{4}\,\mathrm{Ry}$ of H_0. This eigenvalue is four-times degenerate and its eigenspace is spanned by the following functions, given in spherical coordinates

$$\psi_{2lm}(r,\vartheta,\varphi) = C(2,l)\, e^{-r/2}\, L^{2l+1}_{2-l-1}(r)\, r^l\, Y_{lm}(\vartheta,\varphi) , \qquad (l = 0,1\,;\, m = -l,\dots,l) ,$$

where $C(2,l)$ is a number, $L^{2l+1}_{2-l-1}(r)$ is a generalized Laguerre polynomial and $Y_{lm}(\vartheta,\varphi)$ is a spherical harmonic function of degree l and order m. These functions are well-known (cf. [36]) and we can calculate the contribution to the first order energy correction $\langle \psi_{2lm}, V\psi_{2l'm'}\rangle$ for all possible combinations. The result is collected into the following matrix

$$\begin{pmatrix} 0 & \eta & 0 & 0 \\ \overline{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

where

$$\eta := -e\lambda\langle \psi_{200}, V\psi_{210}\rangle \approx 3\, e\, a_0\, \lambda ,$$

and a_0 is the Bohr radius. The eigenvalue problem which is represented by Eq. (3.1) can be solved and its solution is depicted in Figure 3.1. Thus we see that the electric field is responsible for splitting up the four-times degenerate eigenvalue E_2 into three distinct eigenvalues $E_2^{(+)} := E_2 + \eta$, $E_2^{(0)} := E_2$ and $E_2^{(-)} := E_2 - \eta$, where $E_2^{(0)}$ is still two-times degenerate. The corresponding eigenvectors are displayed in Figure 3.1 as well.

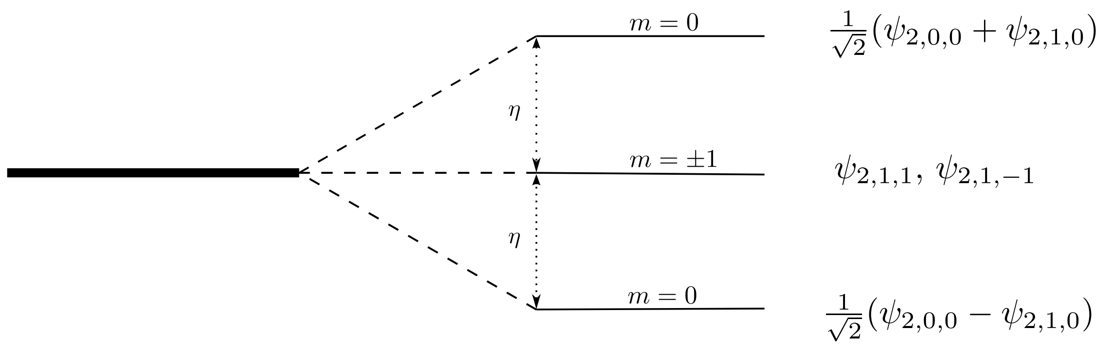


Figure 3.1: Spectral line splitting due to the Stark effect.

The general problem

The example above is a special case of the following abstract spectral problem.
Let H_0 be a linear operator with discret spectrum $\sigma_{\mathrm{disc}}(H_0) = \{E_0, E_1, \dots, E_n\}$, where $E_i \in \mathbb{R}$ for $i \in \{0,\dots,n\}$, and let V be H_0-bounded. Suppose we want to understand the spectrum of the operator

$$H_\lambda = H_0 + \lambda V , \tag{3.2}$$

where $\lambda \in \mathbb{R}$ is the so-called *coupling constant*. Especially, we are interested in the fate of the discrete eigenvalues E_i of H_0 in the spectrum of H_λ for $\lambda > 0$.
The Rayleigh-Schrödinger or regular perturbation theory gives an comprehensive answer to this question. In particular the Kato-Rellich theorem [115, Theorem XII.8], states that for any non-degenerate discrete

eigenvalue E_i of H_0 there exists exactly one point $E_i(\lambda) \in \sigma(H_\lambda)$ near E_i and this point is isolated and non-degenerate. Furthermore, $E_i(\lambda)$ is an analytic function of λ for λ near 0 and there exists an analytic eigenvector $\psi_i(\lambda)$ for λ near 0.
Regular perturbation theory is also applicable in the case of finitely degenerate discrete eigenvalues and a corresponding analyticity result is given in [115, Theorem XII.13].
We mentioned earlier, that usually a Spin-Boson Hamiltonian does not have discrete eigenvalues because the eigenvalues are no longer isolated from the rest of the spectrum. This is due to the free field Hamiltonian H_f and its spectrum $\sigma(H_f) = [0, \infty)$. Thus we can not use regular perturbation theory in cases where the model incorporates a quantized radiation field of massless particles. Such models are called *Pauli-Fierz models* [42].
Much effort was put in proving existence and uniqueness of the ground state for Pauli-Fierz Hamiltonians, see [18, 19, 53, 60, 67, 76, 99, 127] and references therein. Moreover there was much progress in the mathematical analysis of resonances [3, 21, 22, 46, 77, 123].
In addition in various situations it was proved that the ground state and the ground-state energy are analytic functions of the coupling constant [1, 2, 66, 72, 74, 75]. In situations where such an analytic expansion could not be achieved, asymptotic expansions have been proven up to arbitrary order in the coupling constant [12, 24–26, 31, 69].
In general, a lot of effort was put into developing mathematically rigorous results in spectral and scattering theory for various models of quantum field theory. Properties like Compton scattering, Rayleigh scattering, the limiting absorption principle and asymptotic completeness were investigated, see [15, 32–34, 38–40, 47–50, 54–56, 58, 59, 73, 90, 103, 107] and references therein. Note that the references given in this section are not meant to be complete.
Some of the above mentioned results assume that the considered eigenvalues are non-degenerate. In the special case of Spin-Boson models and with regard to results about analyticity of the ground state and ground-state energy we remove the non-degeneracy condition in Chapter 4.

3.2 The operator-theoretic renormalization group method

Many of the statements mentioned at the end of the last section were proved using, at least partially, operator-theoretic renormalization. The back-bone of this method is the Feshbach map. This map, which is also called Schur complement in the finite-dimensional case, is derived from the Feshbach projection method [51]. In the mathematical literature this method has many names. It is for example known as a Grushin Problem [68], as Kreins formula [61, 95, 110] or the Livsic matrix theory [87, 102]. The Feshbach map is a mathematical tool in spectral analysis and singular perturbation theory. In this section we present a variant of it, the so-called *smooth Feshbach-Schur map*. Moreover we outline the *operator-theoretic renormalization group method*.

3.2.1 The Feshbach map

Lets assume we have an operator A on some finite-dimensional Hilbert space $\mathcal{H}_{\text{fin}}$ and orthogonal projections P, $\overline{P} := \mathbb{1} - P$ on $\mathcal{H}_{\text{fin}}$. In the case, that $\overline{P}A\overline{P}$ is invertible on $\overline{P}\mathcal{H}_{\text{fin}}$ we get, due to Schur's block-diagonalization, that A is invertible if and only if its Schur complement is invertible. The Schur complement or Feshbach map is given by

$$F_P(A) := PAP - PA\overline{P}\left(\overline{P}A\overline{P}\right)^{-1}\overline{P}AP\,.$$

This map was used inter alia as a tool in analytic perturbation theory for operators on Hilbert spaces and it was generalized by Bach, Fröhlich and Sigal [19, 20], in the following way.
Let $\mathcal{H}$ be a separable Hilbert space and H_0 be a closed operator that is densely defined on $\mathcal{D} \subset \mathcal{H}$. Choose a bounded projection $P = P^2$ with $\operatorname{Ran}P \subseteq \mathcal{D}$ and such that H_0 commutes with P. We want to stress that P does not need to be orthogonal. Furthermore let W be an H_0-bounded operator defined on $\mathcal{D}$. For $H := H_0 + W$ the Feshbach map is (formally) defined by

$$\begin{aligned} F_P(H) :=& PHP - PH\overline{P}\left(\overline{P}H\overline{P}\right)^{-1}\overline{P}HP \\ =& PH_0 + PWP - PW\overline{P}\left[\overline{P}H_0 + \overline{P}W\overline{P}\right]^{-1}\overline{P}WP\,. \end{aligned} \tag{3.3}$$

Under suitable assumptions $F_P(H)$ is a closed operator on $P\mathcal{H}$ that provides an isospectral map from a class of operators on $\mathcal{H}$ into the operators on $P\mathcal{H} \subseteq \mathcal{H}$. Thus, $F_P(H)$ may pose a simpler spectral problem than the 'full' Hamiltonian H. The non-differentiability of the characteristic function that defines the projection P is a source for potential difficulties [14, Remark 3.9]. Therefore we use the smooth Feshbach(-Schur) map introduced in [14] and generalized in [65].

The smooth Feshbach-Schur map

Let χ and $\overline{\chi}$ be commuting non-zero bounded operators, acting on a separable Hilbert space $\mathcal{H}$ and satisfying $\chi^2 + \overline{\chi}^2 = 1$.

Definition 3.2.1. A *Feshbach pair* (H, T) for χ is a pair of closed operators with the same domain

$$H, T : D(H) = D(T) \subset \mathcal{H} \to \mathcal{H}$$

such that $H, T, W := H - T$, and the operators

$$\begin{aligned} W_\chi &:= \chi W \chi\,, & W_{\overline{\chi}} &:= \overline{\chi} W \overline{\chi}\,, \\ H_\chi &:= T + W_\chi\,, & H_{\overline{\chi}} &:= T + W_{\overline{\chi}}\,, \end{aligned}$$

defined on $D(T)$ satisfy the following assumptions

(a) $\chi T \subset T\chi$ and $\overline{\chi} T \subset T\overline{\chi}$,

(b) $T, H_{\overline{\chi}} : D(T) \subset \mathcal{H} \to \mathcal{H}$ are bijections with bounded inverse.

(c) $\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a bounded operator.

Given a Feshbach pair (H, T) for χ, the operator

$$F_\chi(H, T) := H_\chi - \chi W \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi \tag{3.4}$$

on $D(T)$ is called *Feshbach map of* H. The mapping $(H, T) \mapsto F_\chi(H, T)$ is called *Feshbach map.*

Definition 3.2.2. We call an operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ *bounded invertible* in a subspace $V \subset \mathcal{H}$, if $A : D(A) \cap V \to V$ is a bijection with bounded inverse.

Note that V is not necessarily closed in the above definition. Let (H, T) be a Feshbach pair for χ. We define the following auxiliary operators

$$\begin{aligned} Q_\chi &:= \chi - \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi\,, \\ Q_\chi^\# &:= \chi - \chi W \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi}\,. \end{aligned} \tag{3.5}$$

The importance of these operators becomes apparent in Theorem 3.2.4 but first we take a closer look on some of their properties.

Proposition 3.2.3. *Q_χ and $Q_\chi^\#$ are bounded operators on $D(T)$ and Q_χ leaves $D(T)$ invariant.*

Proof. Since (H, T) is a Feshbach pair for χ the operators H and T are closed on $D(T)$, conditions (a) and (c) in Definition 3.2.1 are valid and by the domain assumptions we have $\mathrm{Ran}\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} \subset D(T)$. Therefore the operators $H(\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi})$ and $T(\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi})$ are well-defined and bounded operators on $D(T)$. Considering $W = H - T$ the operators Q_χ and $Q_\chi^\#$ are also well-defined and bounded on $D(T)$ and Q_χ leaves $D(T)$ invariant. □

Theorem 3.2.4 (Theorem 1, [65])**.** *Let (H, T) be a Feshbach pair for χ on a separable Hilbert space $\mathcal{H}$. Then the following holds*

(a) Let V be a subspace with $\mathrm{Ran}\chi \subset V \subset \mathcal{H}$,

$$T : D(T) \cap V \to V, \quad \text{and} \quad \overline{\chi} T^{-1} \overline{\chi} V \subset V.$$

Then $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ is bounded invertible if and only if $F_\chi(H, T) : D(T) \cap V \to V$ is bounded invertible in V. Moreover,

$$\begin{aligned} H^{-1} &= Q_\chi F_\chi(H, T)^{-1} Q_\chi^\# + \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi}\,, \\ F_\chi(H, T)^{-1} &= \chi H^{-1} \chi + \overline{\chi} T^{-1} \overline{\chi}\,. \end{aligned}$$

(b) $\chi \mathrm{Ker} H \subset \mathrm{Ker} F_\chi(H,T)$ *and* $Q_\chi \mathrm{Ker} F_\chi(H,T) \subset \mathrm{Ker} H$. *The mappings*

$$\chi : \mathrm{Ker} H \to \mathrm{Ker} F_\chi(H,T)\,, \qquad Q_\chi : \mathrm{Ker} F_\chi(H,T) \to \mathrm{Ker} H\,,$$

are linear isomorphisms and inverse to each other.

Remark 3.2.5. Theorem 3.2.4 is a generalization of Theorem 2.1 in [14] to non-self-adjoint χ and $\overline{\chi}$.

Lemma 3.2.6 (Lemma 2, [65])**.** *Let* (H,T) *be a Feshbach pair for* χ *and let* $F := F_\chi(H,T)$, $Q := Q_\chi$, $Q^\# := Q^\#_\chi$ *for simplicity. Then the following identities hold:*

(a) $(\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi}) H = 1 - Q\chi$, *on* $D(T)$, $\qquad H(\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi}) = 1 - \chi Q^\#$, *on* $\mathcal{H}$,

(b) $(\overline{\chi} T^{-1} \overline{\chi}) F = 1 - \chi Q$, *on* $D(T)$, $\qquad F(\overline{\chi} T^{-1} \overline{\chi}) = 1 - Q^\# \chi$, *on* $\mathcal{H}$,

(c) $HQ = \chi F$, *on* $D(T)$, $\qquad Q^\# H = F\chi$, *on* $D(T)$.

Lemma 3.2.7 (Lemma 3, [65])**.** *Conditions (a), (b) and (c) on Feshbach pairs are satisfied if*

(a') $\chi T \subset T\chi$ *and* $\overline{\chi} T \subset T \overline{\chi}$,

(b') T *is bounded invertible on* $\mathrm{Ran}\overline{\chi}$,

(c') $\|T^{-1}\overline{\chi} W \overline{\chi}\| < 1$ *and* $\|\overline{\chi} W T^{-1} \overline{\chi}\| < 1$.

Remark 3.2.8. The special case, where $\chi = \chi^2$ and $\overline{\chi} = \mathbb{1} - \chi$ are projections is exactly the original Feshbach map or Feshbach projection method used in [19,20] and stated in Eq. (3.3).

3.2.2 The operator-theoretic renormalization group method

In the following we want to understand the spectral behavior of a closed, possibly unbounded operator $H := T + W$ on some Hilbert space $\mathcal{H}$. We assume that we have complete knowledge of the spectrum of the closed operator T and that W is a perturbation of T defined on the entire domain of T. Moreover we assume that (H,T) is a Feshbach pair for some bounded operator χ on $\mathcal{H}$. We set $\overline{\chi} = \sqrt{1-\chi^2}$. Using the isospectrality property of the smooth Feshbach map

$$z \in \sigma\big(H\big) \quad \Longleftrightarrow \quad 0 \in \sigma\big(H - z\big) \quad \Longleftrightarrow \quad 0 \in \sigma\big(F_\chi(H - z, T - z)\big)\,, \tag{3.6}$$

which follows directly from Theorem 3.2.4, we can analyse the spectrum of H by studying the spectrum of the Feshbach map

$$F_\chi(H - z, T - z) = T - z + W_\chi - \chi W \overline{\chi} (H - z)_{\overline{\chi}}^{-1} \overline{\chi} W \chi\,.$$

By choosing a suitable operator χ this can be a much more easier task. For example if $\mathrm{Ran}\chi$ is finite dimensional. Furthermore, under certain circumstances, we can iterate this procedure and get a limiting Hamiltonian with a spectrum that is even simpler to analyse. Roughly said, this will be the case if the relative size of the perturbation W decreases on every step compared to the iterated version of the unperturbed operator T. We refer to Remark 3.3.8 and the related subsections in Chapter 4 for more details. In context with the Feshbach map the following well-known theorem by Carl Neumann (1832-1925) is often very useful. It can be used to expand the Feshbach map into an absolutely convergent power series. Neumann proved a similar theorem for so-called *Bessel functions* in [109]. The theorem below is stated for a bounded operator on some Banach space. Note that we slightly changed the notation in contrast to the referenced version.

Theorem 3.2.9 (Neumann's Theorem, [136])**.** *Let* $T : \mathfrak{B} \to \mathfrak{B}$ *be a bounded linear operator on a Banach space* $\mathfrak{B}$. *Suppose that* $\|T\| < 1$. *Then* $\mathbb{1} - T$ *has a unique bounded linear inverse* $(\mathbb{1} - T)^{-1}$ *which is given by the Neumann series*

$$(\mathbb{1} - T)^{-1} = \sum_{k=0}^{\infty} T^k\,. \tag{3.7}$$

This series is absolutely convergent in the sense that $\lim_{N\to\infty} \sum_{k=0}^{N} T^k$ *converges in norm to* $(\mathbb{1} - T)^{-1}$.

The general strategy of the operator-theoretic renormalization group method

In the following we summaries the *operator-theoretic renormalization group method* based on the smooth Feshbach-Schur map. In this form it was introduced by Bach et al. in [14].
Let $\mathcal{H} := \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}$ be a Hilbert space, where $\mathcal{H}_{\mathrm{at}}$ and $\mathcal{F}$ are defined as in Chapter 2. Let H, T be closed operators on $\mathcal{H}$ and assume that $W := H - T$ induces an interaction between the elements of the atomic space $\mathcal{H}_{\mathrm{at}}$ and the Fock space $\mathcal{F}$.
The operator-theoretic renormalization group method consists of the following steps:

1. Prove that H and T are a Feshbach pair for a suitable non-zero bounded operator χ_ρ, $0 < \rho \leq 1$.
2. Construction of an effective Hamiltonian $H_g^{(1,\rho)}(z)$ (Initial Feshbach Step).
   i) decimate degrees of freedom in the atomic space, i.e. apply a projection onto the eigenspace of T corresponding to the eigenvalue for which we want to know the behavior under perturbation.
   ii) decimation of photon degrees of freedom, i.e. restrict the Fock space to the spectral subspace corresponding to photon energies less than ρ.

   These two decimations are preformed using the Feshbach map $F_{\chi_\rho}(H,T)$ with suitably chosen operator χ_ρ, since it maps operators on a given space (the space on which χ_ρ is defined) into operators on $\mathrm{Ran}\chi_\rho$.
   Remark. The first and second step are intertwined since at some point we have to prove that (H,T) are a Feshbach pair for χ_ρ in order to justify the use of the Feshbach map in the construction of the effective Hamiltonian.
3. Construction of a Banach space of integral kernels $(w_{m,n})_{m,n\in\mathbb{N}}$ on which operators of the following form are defined
   $$T[H_f] + W - E \cdot \mathbb{1}_{\mathcal{H}_{\mathrm{red}}}\,,$$
   where T is a continuously differentiable function with $T[0] = 0$ and $T[H_f] \in \mathcal{L}(\mathcal{H}_{\mathrm{red}})$,
   $$W := \sum_{m+n\geq 1} H_{m,n}$$
   with $H_{m,n}(w_{m,n}) \in \mathcal{L}(\mathcal{H}_{\mathrm{red}})$ and $E \in \mathbb{C}$.
   Remark. The reduced Fock space $\mathcal{H}_{\mathrm{red}}$ is defined in Eq. (2.12). In Section 3.3 we construct the Banach spaces of integral kernels that we use in Chapter 4.
4. Prove that the effective Hamiltonian $H_g^{(1,\rho)}(z)$ is in a neighborhood of the free field Hamiltonian H_f (Banach space estimate for the first Step).
   Remark. This neighborhood is given in terms of a norm defined on the Banach spaces of integral kernels constructed in the third step. More details are given in the related subsections in Chapter 4.
5. Iterate this procedure by applying the renormalization map $\mathcal{R}_\rho$.
   The map $\mathcal{R}_\rho$ is, loosely speaking, a composition of the Feshbach map and a scaling transformation.
   Remark. Since we are already in a neighborhood of the free field Hamiltonian due to step 4 we can iterate the Feshbach method such that the iterated Hamiltonians $H_g^{(n+1,\rho)}(z) = \mathcal{R}_\rho^n H_g^{(1,\rho)}(z)$ stay in this neighborhood and ideally converge isospectral to a limiting Hamiltonian. Detailed explanations when this happens and an exact definition of the map $\mathcal{R}_\rho$ are given in the corresponding subsections in Chapter 4.
6. Analyse the spectrum of the limiting Hamiltonian and extract useful informations on the spectrum of the original Hamiltonian H.

Remark 3.2.10. This summary can be used as a guide through the sections in Chapter 4. For a more thorough introduction to the operator-theoretic renormalization group method we refer the reader to [14].

3.3 Operator-valued integral kernels

In this section we construct Banach spaces of integral kernels that we use in Chapter 4 to control the renormalization transformation. Moreover, we give a precise meaning to field operators defined by operator-valued integral kernels and define, for measurable operator-valued functions, a particular sesquilinear form that determines uniquely a bounded linear operator. Furthermore, we state a generalized version of Wick's theorem and review some very useful technical auxiliaries.

3.3.1 Banach spaces of integral kernels

In order to control the renormalization transformation, in particular proving its convergence, we introduce Banach spaces of integral kernels. Since we want to treat degenerate situations in Chapter 4 we extend the notation of integral kernels to matrix-valued integral kernels. Specifically, the Banach spaces which we introduce below are a straightforward generalization of the spaces defined in [14] or [66]. A generalization to matrix-valued integral kernels seems to be a canonical choice to accommodate degenerate situations. We note that the choice of these spaces is not unique.

For $d \in \mathbb{N}$ we define the Banach space $\mathcal{W}_{0,0}^{[d]}$ as the space of continuously differentiable matrix-valued functions

$$\mathcal{W}_{0,0}^{[d]} := C^1([0,1]; \mathcal{L}(\mathbb{C}^d))$$

with norm

$$\|w\|_{C^1} := \|w\|_\infty + \|w'\|_\infty \,,$$

where $(\cdot)'$ stands for the derivative. Let $\boldsymbol{B}_1 := \{\boldsymbol{k} \in \mathbb{R}^3 : |\boldsymbol{k}| \leq 1\}$. For a set $\boldsymbol{A} \subset \mathbb{R}^3$ we write

$$A := \boldsymbol{A} \times \{1,2\} \,, \qquad \int_A dk := \sum_{\lambda=1,2} \int_{\boldsymbol{A}} d^3\boldsymbol{k} \,,$$

where we recall the notation of Eq. (2.5). Moreover for $m, n \in \mathbb{N}$ we write

$$\begin{aligned} k^{(m)} &:= (k_1, ..., k_m) \in (\mathbb{R}^3 \times \{1,2\})^m \,, \\ \tilde{k}^{(n)} &:= (\tilde{k}_1, ..., \tilde{k}_n) \in (\mathbb{R}^3 \times \{1,2\})^n \,, \\ K^{(m,n)} &:= (k^{(m)}, \tilde{k}^{(n)}) \,. \end{aligned}$$

$$\begin{aligned} dk^{(m)} &:= dk_1 \cdots dk_m \,, \\ d\tilde{k}^{(n)} &:= d\tilde{k}_1 \cdots d\tilde{k}_n \,, \\ dK^{(m,n)} &:= dk^{(m)} d\tilde{k}^{(n)} \,. \end{aligned}$$

$$\begin{aligned} |k^{(m)}| &:= |k_1| \cdots |k_m| \,, \\ |\tilde{k}^{(n)}| &:= |\tilde{k}_1| \cdots |\tilde{k}_n| \,, \\ |K^{(m,n)}| &:= |k^{(m)}||\tilde{k}^{(n)}| \,. \end{aligned}$$

Furthermore, we shall use

$$\Sigma[k^{(n)}] := \sum_{i=1}^n |k_i| \,, \qquad \Sigma[\tilde{k}^{(m)}] := \sum_{i=1}^m |\tilde{k}_i| \,.$$

For $m, n \in \mathbb{N}$ with $m + n \geq 1$ and $\mu > 0$ we denote by $\mathcal{W}_{m,n}^{[d]}$ the space of measurable functions $w_{m,n} : B_1^{m+n} \to \mathcal{W}_{0,0}^{[d]}$ satisfying the following three properties.

(i) the $w_{m,n}$ are symmetric with respect to all permutations of the m arguments from B_1^m and the n arguments from B_1^n, respectively.

(ii) for $m + n \geq 1$, we have $w_{m,n}(k^{(m)}, \tilde{k}^{(n)})(r) = 0$, provided $r \geq 1 - \max(\Sigma[k^{(m)}], \Sigma[\tilde{k}^{(n)}])$.

(iii) the following norm is finite

$$\|w_{m,n}\|_\mu^\# := \|w_{m,n}\|_\mu + \|\partial_r w_{m,n}\|_\mu \,,$$

where

$$\|w_{m,n}\|_\mu := \left(\int_{B_1^{m+n}} \|w_{m,n}(K^{(m,n)})\|_\infty^2 \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} \right)^{1/2} .$$

Remark 3.3.1. The space $\mathcal{W}_{m,n}^{[d]}$ is given as the subspace of

$$L^2 \left(B^{m+n}, \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}}; \mathcal{W}_{0,0}^{[d]} \right) , \tag{3.8}$$

consisting of elements satisfying Conditions (i) and (ii) above. Note that the norm $\|\cdot\|_\mu^\#$ is equivalent to the natural norm of (3.8) (given by the theory of Banach space-valued L^p-functions), which is the norm chosen in [66]. Moreover, we identify (3.8) as a subspace of $L^2\left([0,1]\times B^{m+n}, \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}}; \mathcal{L}(\mathbb{C}^d)\right)$ by means of

$$w_{m,n}(r,k^{(m)},\tilde{k}^{(n)}) = w_{m,n}(k^{(m)},\tilde{k}^{(n)})(r)\,. \tag{3.9}$$

Henceforth we use this identification without further comment. Furthermore, we remark that

$$\|w_{0,0}\|_{C^1} = \|w_{0,0}\|_\mu^\#\,,$$

with the natural convention that for $m = n = 0$ the empty Cartesian product consists of a single point and that there is no integration in that case.

For given $\xi \in (0,1)$ and $\mu > 0$ we define the Banach space

$$\mathcal{W}_\xi^{[d]} := \bigoplus_{m,n\in\mathbb{N}_0} \mathcal{W}_{m,n}^{[d]}\,,$$

with norm

$$\|w\|_{\mu,\xi}^\# := \|w_{0,0}\|_{C^1} + \sum_{m,n\geq 1} \xi^{-(m+n)} \|w_{m,n}\|_\mu^\#\,,$$

for $w = (w_{m,n})_{m,n\in\mathbb{N}_0} \in \mathcal{W}_\xi^{[d]}$.

Next we define a linear mapping $H : \mathcal{W}_\xi^{[d]} \to \mathcal{L}(\mathcal{H}_{\mathrm{red}})$. In order to do this we use the notation

$$a^*(k^{(m)}) := \prod_{i=1}^m a^*(k_i)\,, \qquad a(\tilde{k}^{(n)}) := \prod_{i=1}^n a(\tilde{k}_i)\,.$$

If $w \in \mathcal{W}_{0,0}^{[d]}$ we define $H_{0,0}(w) := w_{0,0}(H_f)$. For $m+n \geq 1$ and $w_{m,n} \in \mathcal{W}_{m,n}^{[d]}$ we define the following operator on $\mathcal{H}_{\mathrm{red}}$

$$H_{m,n}(w_{m,n}) := P_{\mathrm{red}} \left(\int_{B_1^{m+n}} a^*(k^{(m)})\, w_{m,n}(H_f, K^{(m,n)})\, a(\tilde{k}^{(n)})\, \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \right) P_{\mathrm{red}}\,. \tag{3.10}$$

The subsequent lemma is proven in [14, Theorem 3.1].

Lemma 3.3.2. *Let $\mu > 0$ and $m,n \in \mathbb{N}_0$ with $m+n \geq 1$. For $w_{m,n} \in \mathcal{W}_{m,n}^{[d]}$ we have*

$$\|H_{m,n}(w_{m,n})\| \leq \frac{\|w_{m,n}\|_\mu}{\sqrt{n^n m^m}}\,. \tag{3.11}$$

Using our notation above and the convention that $p^p := 1$ for $p = 0$ it is trivial to extend this lemma to the case $m+n = 0$.

For sequences $w = (w_{m,n})_{(m,n)\in\mathbb{N}_0^2} \in \mathcal{W}_\xi^{[d]}$ we define the operator $H(w)$ by the sum

$$H(w) := \sum_{m,n} H_{m,n}(w_{m,n})\,, \tag{3.12}$$

where the sum converges in operator norm, which can be seen using Eq. (3.11).

Remark 3.3.3. The right hand side of Eq. (3.12) is said to be in *Wick-ordered* form. We note that this is also called the *generalized normal-ordered form* of the operator $H(w)$.

A proof of the following theorem can be found in [14] with a modification explained in [71].

Theorem 3.3.4. *Let $\mu > 0$ and $0 < \xi < 1$. Then the map $H : \mathcal{W}_\xi^{[d]} \to \mathcal{L}(\mathcal{H}_{\mathrm{red}})$ is injective and bounded. For $w \in \mathcal{W}_\xi^{[d]}$ and for $\tilde{w} \in \mathcal{W}_\xi^{[d]}$ with $\tilde{w}_{0,0} = 0$ we have*

$$\|H(w)\| \leq \|w\|_{\mu,\xi}^\#\,, \qquad \|H(\tilde{w})\| \leq \xi \|\tilde{w}\|_{\mu,\xi}^\#\,. \tag{3.13}$$

At the end of this subsection we want to examine the renormalization process described in the fifth step of the operator-theoretic renormalization group method. Such a renormalization involves a rescaling of the energy. This rescaling is described by means of a dilation operator, which we shall now define.

Definition 3.3.5. Let $\rho > 0$. We define the *operator of dilation* on the one particle sector by

$$U_\rho : \mathfrak{h} \to \mathfrak{h}, \quad (U_\rho \varphi)(k) = \rho^{3/2} \varphi(\rho k),$$

where we use the notation $\rho k := (\rho \boldsymbol{k}, \lambda)$. We define the operator of dilation on Fock space by

$$\Gamma_\rho = \bigoplus_{n=0}^{\infty} U_\rho^{\otimes n} \restriction \mathcal{F}.$$

We define the mapping $S_\rho : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{F})$ called *rescaling by dilation* by

$$S_\rho(A) := \rho^{-1} \Gamma_\rho(A) \Gamma_\rho^*. \tag{3.14}$$

Remark 3.3.6. We note that by definition $\Gamma_\rho \Omega = \Omega$. Moreover, one can show that

$$\Gamma_\rho \, a^*(k) \, \Gamma_\rho^* = \rho^{-3/2} a^*(\rho^{-1} k), \quad \Gamma_\rho \, a(k) \, \Gamma_\rho^* = \rho^{-3/2} a(\rho^{-1} k).$$

The subsequent lemma relates the scaling transformation to a scaling transformation of the integral kernels. It is straightforward to verify using the substitution formula.

Lemma 3.3.7. *For $w \in \mathcal{W}_\xi^{[d]}$ define the scaling transformation of the integral kernel by*

$$s_\rho(w_{m,n})[r, K^{(m,n)}] := \rho^{(m+n)-1} w_{m,n}[\rho r, \rho K^{(m,n)}].$$

Then

$$S_\rho(H(w)) = H(s_\rho(w)).$$

Remark 3.3.8. For $m + n \geq 1$ one finds

$$\|s_\rho(w_{m,n})\|_\mu \leq \rho^{\mu(m+n)} \|w_{m,n}\|_\mu.$$

This illustrates, that in every renormalization step the relative size $\|w_{m,n}\|_\mu$ of the perturbative part, $m+n \geq 1$, of an integral kernel $w \in \mathcal{W}_\xi^{[d]}$ shrinks compared to the size of its unperturbed part $\|w_{0,0}\|_{C^1}$.

3.3.2 Field operators, Pull-Through Formula and Wick's theorem

We consider the Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}$ consisting of a separable Hilbert space $\mathcal{H}_{\mathrm{at}}$ and the bosonic Fock Space $\mathcal{F}$ as defined in Chapter 2.

Field Operators Associated to Integral Kernels

In the following we give a precise meaning to field operators defined by operator-valued integral kernels. Thereby we extend the definition of smeared annihilation and creation operators given in Section 2.2. Let $X := \mathbb{R}^3 \times \mathbb{Z}_2$ and $k_1, \ldots, k_{m+n} \in X$. For $\psi \in \mathcal{F}_0$ we have by Eq. (2.8),

$$[a(k_1) \cdots a(k_m) \psi]^{(n)} (k_{m+1}, ..., k_{m+n}) = \sqrt{\frac{(m+n)!}{n!}} \psi^{(m+n)}(k_1, ..., k_{m+n}). \tag{3.15}$$

Moreover, using Fubini's theorem [117, Theorem I.21], it is elementary to see that the vector-valued map $(k_1, ..., k_m) \mapsto a(k_1) \cdots a(k_m) \psi$ is an element of $L^2(X^m; \mathcal{F})$. For measurable functions $w_{m,n}$ on $(\mathbb{R}^3 \times \mathbb{Z}_2)^{n+m}$ with values in the linear operators of $\mathcal{H}_{\mathrm{at}}$ we define the sesquilinear form

$$\int_{(\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n}} \left\langle a(k^{(m)}) \varphi, \, w_{m,n}(K^{(m,n)}) \, a(\tilde{k}^{(n)}) \psi \right\rangle \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}}.$$

This form is defined for all φ and ψ in $\mathcal{H}$, for which the integrand on the right hand side is integrable. By the Riesz lemma [117, Theorem II.4], we obtain a densely defined linear operator, which can easily be shown to be closable. We denote the closure of this operator by $H^{(0)}(w_{m,n})$. By adjusting the notation this provides a precise definition of annihilation and creation operators for so-called *coupling functions* $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$. In particular we define the smeared annihilation and creation operators by

$$a(G) := \int G^*(k) a(k) dk, \qquad a^*(G) := \int G(k) a^*(k) dk. \tag{3.16}$$

Note that this identification is similar to the one in Eq. (2.9). Likewise these operators satisfy the canonical commutation relations (Eq. (2.4)). An abstract definition of these operators is given in the subsequent remark.

Remark 3.3.9. An element G of $L^2(\mathbb{R}^3\times\mathbb{Z}_2;\mathcal{L}(\mathcal{H}_{\mathrm{at}}))$ defines a linear operator $G:\mathcal{H}_{\mathrm{at}}\to L^2(\mathbb{R}^3\times\mathbb{Z}_2;\mathcal{H}_{\mathrm{at}})$ by

$$(G\varphi)(k):=G(k)\varphi,\quad \text{for } \varphi\in\mathcal{H}_{\mathrm{at}}.$$

This operator is bounded with $\|G\|\le\|G\|_2$. Since $L^2(\mathbb{R}^3\times\mathbb{Z}_2;\mathcal{H}_{\mathrm{at}})\cong\mathcal{H}_{\mathrm{at}}\otimes\mathfrak{h}$, we can consider G as an element of $\mathcal{L}(\mathcal{H}_{\mathrm{at}},\mathcal{H}_{\mathrm{at}}\otimes\mathfrak{h})$ and hence $L^2(\mathbb{R}^3\times\mathbb{Z}_2;\mathcal{L}(\mathcal{H}_{\mathrm{at}}))$ as a subspace embedded in $\mathcal{L}(\mathcal{H}_{\mathrm{at}},\mathcal{H}_{\mathrm{at}}\otimes\mathfrak{h})$. For an operator $G\in\mathcal{L}(\mathcal{H}_{\mathrm{at}},\mathcal{H}_{\mathrm{at}}\otimes\mathfrak{h})$ and vectors $\varphi\in\mathcal{H}_{\mathrm{at}}$, $\psi\in S_{n-1}(\otimes^{n-1}\mathfrak{h})$ the smeared creation operator $a^*(G)$ is defined as the closure in $\mathcal{H}$ of the linear operator given by

$$a^*(G)(\varphi\otimes\psi):=\sqrt{n}S_n(G\varphi\otimes\psi),$$

where the projection operator S_n was defined in Eq. (2.2). The annihilation operator $a(G)$ is defined as the adjoint of $a^*(G)$.

In order to define field operators that depend on the free field energy we consider measurable functions $w_{m,n}$ on $\mathbb{R}_+\times(\mathbb{R}^3\times\mathbb{Z}_2)^{n+m}$ with values in the linear operators of $\mathcal{H}_{\mathrm{at}}$. To such a function we associate the sesquilinear form

$$q_{w_{m,n}}(\varphi,\psi):=\int\limits_{(\mathbb{R}^3\times\mathbb{Z}_2)^{m+n}}\left\langle a(k^{(m)})\varphi,\, w_{m,n}(H_f,K^{(m,n)})\,a(\tilde{k}^{(n)})\psi\right\rangle\frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}},$$

defined for all φ and ψ in $\mathcal{H}$, for which the integrand on the right hand side is integrable. If the integral kernel decays sufficiently fast as a function of the free field energy, the sesquilinear form defines a bounded operator. To show this we use the following lemma and the identification in Eq. (3.9).

For a locally compact space Y we denote by $C_\infty(Y)$ the algebra of continuous functions vanishing at infinity. That is the set of continuous functions $f\in C(Y)$ with the property that for any $\epsilon>0$, there is a compact set $D_\epsilon\subset Y$ such that $|f(x)|<\epsilon$ if $x\notin D_\epsilon$.

Lemma 3.3.10. *For measurable $w:X^{m+n}\to C_\infty([0,\infty))$, we define*

$$\|w_{m,n}\|_\sharp^2:=\int_{X^{m+n}}\sup_{r\ge0}\left[\|w_{m,n}(r,K^{(m,n)})\|^2\prod_{l=1}^{m}\{r+\Sigma[k^{(l)}]\}\prod_{\tilde{l}=1}^{n}\{r+\Sigma[\tilde{k}^{(\tilde{l})}]\}\right]\frac{dK^{(m,n)}}{|K^{(m,n)}|^2}.$$

Then for all $\varphi,\psi\in\mathcal{H}$ with finitely many particles

$$|\,q_{w_{m,n}}(\varphi,\psi)\,|\le\|w_{m,n}\|_\sharp\,\|\varphi\|\,\|\psi\|\,. \tag{3.17}$$

In particular, if $\|w_{m,n}\|_\sharp<\infty$, the form $q_{w_{m,n}}$ determines uniquely a bounded linear operator $H_{m,n}(w_{m,n})$ such that

$$q_{w_{m,n}}(\varphi,\psi)=\langle\varphi,\,H^{(0)}_{m,n}(w_{m,n})\psi\rangle\,,$$

for all φ,ψ in $\mathcal{H}$. Moreover we have $\|H^{(0)}_{m,n}(w_{m,n})\|\le\|w_{m,n}\|_\sharp$.

Proof. We set $P[k^{(n)}]:=\prod_{l=1}^n(H_f+\Sigma[k^{(l)}])^{1/2}$ and insert $\mathbb{1}$'s into the left hand side of Eq. (3.17) to obtain the trivial identity

$$\begin{aligned}|\,q_{w_{m,n}}(\varphi,\psi)\,|=\Bigg|\int_{X^{m+n}}\Big\langle P[k^{(m)}]P[k^{(m)}]^{-1}|k^{(m)}|^{1/2}a(k^{(m)})\varphi,w_{m,n}(H_f,K^{(m,n)})\\ P[\tilde{k}^{(n)}]P[\tilde{k}^{(n)}]^{-1}|\tilde{k}^{(n)}|^{1/2}a(\tilde{k}^{(n)})\psi\Big\rangle\frac{dK^{(m,n)}}{|K^{(m,n)}|}\Bigg|.\end{aligned}$$

The lemma now follows using the Cauchy-Schwarz inequality and the following well-known identity for $n\ge1$ and $\phi\in\mathcal{F}$,

$$\int_{X^n}\left|k^{(n)}\right|\left\|\prod_{l=1}^{n}\left[H_f+\Sigma[k^{(l)}]\right]^{-1/2}a(k^{(n)})\phi\right\|^2dk^{(n)}=\left\|P_\Omega^\perp\phi\right\|^2, \tag{3.18}$$

where $P_\Omega^\perp:=\mathbb{1}-|\Omega\rangle\langle\Omega|$ and the symbol $|\Omega\rangle\langle\Omega|$ denotes the orthogonal projection onto the vacuum vector. A proof of Eq. (3.18) is given in [74, Appendix A]. The last statement of the lemma follows from the first and the Riesz lemma [117, Theorem II.4]. □

The Pull-Through Formula and elementary estimates

The subsequent lemma states the well-known *Pull-Through Formula*. It can be proved using Eq. (3.15). For a detailed proof we refer to [20, 74].

Lemma 3.3.11. *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a bounded measurable function. Then for all $k \in \mathbb{R}^3 \times \mathbb{Z}_2$*

$$f(H_f)\, a^*(k) = a^*(k)\, f(H_f + \omega(k)), \qquad a(k)\, f(H_f) = f(H_f + \omega(k))\, a(k).$$

In Chapter 4 we use the estimates from the following two lemmas on multiple occasions. They establish elementary estimates for the annihilation and creation operators defined in Eq. (3.16).

Lemma 3.3.12. *For $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$ we have*

$$\begin{aligned} \| a(G)\, H_f^{-1/2} \| &\le \| \omega^{-1/2} G \|, \\ \| a^*(G)\, (H_f + 1)^{-1/2} \| &\le \| (\omega^{-1} + 1)^{1/2} G \|. \end{aligned} \tag{3.19}$$

A proof of this lemma is given in [19]. Since the proof nicely illustrates how the smeared annihilation and creation operators can be estimated by the free field Hamiltonian we give a complete proof nevertheless.

Proof. We use the notation of Eq. (2.5). We set $N := \int a^*(k) a(k) dk$ and let $\psi \in \mathbb{1}_{N \le n}\mathcal{H}$ for some $n \in \mathbb{N}$. In order to prove the first inequality we estimate

$$\begin{aligned} \|a(G)\psi\| &\le \int \|G(k)\, a(k)\psi\|\, dk \\ &= \int \|G(k)\, |k|^{-1/2} |k|^{1/2}\, a(k)\psi\|\, dk \\ &\le \Big(\int |k|\, \|a(k)\psi\|^2 dk\Big)^{1/2} \Big(\int |k|^{-1} \|G(k)\|^2 dk\Big)^{1/2} \\ &= \Big(\int |k|^{-1} \|G(k)\|^2 dk\Big)^{1/2} \|H_f^{1/2}\psi\|. \end{aligned}$$

To prove the second inequality we use the commutation relations

$$\begin{aligned} \|a^*(G)\psi\|^2 &= \langle a^*(G)\psi,\, a^*(G)\psi\rangle = \langle \psi,\, a(G)\, a^*(G)\psi\rangle \\ &= \langle \psi,\, \big(a^*(G)\, a(G) + \int \|G(k)\|^2 dk\big)\psi\rangle \\ &\le \Big(\int |k|^{-1} \|G(k)\|^2 dk\Big) \|H_f^{1/2}\psi\|^2 + \int \|G(k)\|^2 dk \|\psi\|^2. \end{aligned}$$ □

Lemma 3.3.13. *For $G \in L^2\big(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}})\big)$ and $r \in \mathbb{R}_+$ we have the following estimate*

$$\|a(G)\mathbb{1}_{H_f \le r}\| \le \Big(\int_{|k| \le r} \frac{\|G(k)\|^2}{|k|} dk\Big)^{1/2} r^{1/2}. \tag{3.20}$$

Proof. We obtain this through the following estimate

$$\begin{aligned} \|a(G)\mathbb{1}_{H_f \le r}\| &= \|a\big(G\, \mathbb{1}_{|k| \le r}\big)\, H_f^{-1/2} H_f^{1/2}\, \mathbb{1}_{H_f \le r}\| \\ &\le \|a(G\, \mathbb{1}_{|k| \le r})\, H_f^{-1/2}\| \|H_f^{1/2}\, \mathbb{1}_{H_f \le r}\| \\ &\le \Big(\int_{|k| \le r} \frac{\|G(k)\|^2}{|k|} dk\Big)^{1/2} r^{1/2}. \end{aligned}$$ □

Generalized Wick's theorem

For $m, n \in \mathbb{N}_0$ let $\mathcal{M}_{m,n}$ denote the space of measurable functions on $\mathbb{R}_+ \times (\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n}$ with values in the linear operators on $\mathcal{H}_{\mathrm{at}}$. Let

$$\mathcal{M} = \bigoplus_{m+n=1} \mathcal{M}_{m,n}.$$

Then we define for $w \in \mathcal{M}$ the operator

$$W[w] := \sum_{m+n=1} H^{(0)}_{m,n}(w)\,. \tag{3.21}$$

In the case that $w \in \mathcal{W}^{[d]}_{\xi}$ we define, according to (3.12),

$$W[w] := \sum_{m,n\in\mathbb{N}_0} H_{m,n}(w)\,. \tag{3.22}$$

The following theorem is from [20]. It is a generalization of Wick's theorem [133].

Theorem 3.3.14 (Generalized Wick's theorem)**.** *Let $w \in \mathcal{M}$ or $w \in \mathcal{W}^{[d]}_{\xi}$ and let $F_0, F_1, ..., F_L \in \mathcal{M}_{0,0}$ resp. $F_0, F_1, ..., F_L \in \mathcal{W}^{[d]}_{0,0}$. Then as a formal identity*

$$F_0(H_f)W[w]F_1(H_f)W[w]\cdots W[w]F_{L-1}(H_f)W[w]F_L(H_f) = H(\widetilde{w}^{(\mathrm{sym})})\,,$$

where $\widetilde{w}^{(\mathrm{sym})}$ is the symmetrization with respect to $k^{(M)}$ and $\widetilde{k}^{(N)}$ of

$$\begin{aligned}\widetilde{w}_{M,N}(r;K^{(M,N)}) = &\sum_{\substack{m_1+...+m_L=M\\ n_1+...+n_L=N}} \ \sum_{\substack{p_1,q_1...,p_L,q_L:\\ m_l+p_l+n_l+q_l\geq 0}} \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l} \right\}\\ &\times F_0(r+\tilde{r}_0)\langle\Omega, \prod_{l=1}^{L-1}\left\{ W^{m_l,n_l}_{p_l,q_l}[w](r+r_l;K^{(m_l,n_l)}_l)F_l(H_f+r+\widetilde{r}_l)\right\}\\ &\times W^{m_L,n_L}_{p_L,q_L}[w](r+r_L;K^{(m_L,n_L)}_L)\Omega\rangle F_L(r+\widetilde{r}_L)\,,\end{aligned}$$

with

$$\begin{aligned}&K^{(M,N)} := (K^{(m_1,n_1)}_1,\ldots,K^{(m_L,n_L)}_L)\,, \qquad K^{(m_l,n_l)}_l := (k^{(m_l)}_l,\widetilde{k}^{(n_l)}_l)\,,\\ &r_l := \Sigma[\widetilde{k}^{(n_1)}_1]+\cdots+\Sigma[\widetilde{k}^{(n_{l-1})}_{l-1}]+\Sigma[k^{(m_{l+1})}_{l+1}]+\cdots+\Sigma[k^{(m_L)}_L]\,,\\ &\widetilde{r}_l := \Sigma[\widetilde{k}^{(n_1)}_1]+\cdots+\Sigma[\widetilde{k}^{(n_l)}_l]+\Sigma[k^{(m_{l+1})}_{l+1}]+\cdots+\Sigma[k^{(m_L)}_L]\,.\end{aligned} \tag{3.23}$$

Moreover, if $W[w]$ is given by Eq. (3.21) we have

$$W^{m_l,n_l}_{p_l,q_l}[w](\,\cdot\,;K^{(m_l,n_l)}_l) = \int\limits_{(\mathbb{R}^3\times\mathbb{Z}_2)^{p_l+q_l}} a^*(x^{(p_l)})\, w_{m_l+p_l,n_l+q_l}[k^{(m_l)},x^{(p_l)},\tilde{k}^{(n_l)},\tilde{x}^{(q_l)}]\, a(\tilde{x}^{(q_l)})\,\frac{dX^{(p_l,q_l)}}{|X^{(p_l,q_l)}|^{1/2}}\,,$$

and in case of Eq. (3.22) we have

$$\begin{aligned}&W^{m_l,n_l}_{p_l,q_l}[w](r;K^{(m_l,n_l)}_l)\\ &\quad= \mathbb{1}_{[0,1]}(H_f)\int\limits_{B_1^{p_l+q_l}} a^*(x^{(p_l)})\, w_{m_l+p_l,n_l+q_l}[H_f+r,k^{(m_l)},x^{(p_l)},\tilde{k}^{(n_l)},\tilde{x}^{(q_l)}]\, a(\tilde{x}^{(q_l)})\,\frac{dX^{(p_l,q_l)}}{|X^{(p_l,q_l)}|^{1/2}}\;\mathbb{1}_{[0,1]}(H_f)\,.\end{aligned}$$

A proof of the generalized Wick's theorem is given in [20]. We note that the proof is essentially the same as the proofs of Theorem 3.6 in [14] and Theorem 7.2 in [74]. The above choices for $W^{m_l,n_l}_{p_l,q_l}[w](r;K^{(m_l,n_l)}_l)$ become reasonable in Chapter 4.

4 Degenerate perturbation theory

Low-energy phenomena of quantum mechanical matter interacting with a quantized field of massless particles have been mathematically investigated extensively. We refer the interested reader to [128] to get a read on this. Moreover there exists extensive, mathematically rigorous literature on physical properties like the existence of ground state and resonances, dispersion relations, asymptotic completeness and much more. We refer the reader to Chapter 3 for a collection of references. Some of these results were proved using the method of operator-theoretic renormalization (Subsection 3.2). However, the application of this method usually requires that the unperturbed eigenstate is non-degenerate, i.e. the projection onto the corresponding eigenspace is a rank-one operator. Especially analyticity results for the ground state and resonances make use of this assumption. In this chapter we consider cases where we permit degeneracy. In particular we look at two distinct cases. A degeneracy can be caused by a set of symmetries which act irreducibly on the eigenspace of the ground-state eigenvalue. H.A. Kramers discovered this kind of degeneracy in the energy levels of a quantum mechanical systems with an odd total number of fermions [94]. If the irreducibility assumptions are not met, the eigenspace of the degenerate eigenvalue is expected to split up at higher order in perturbation theory. This is motivated by physics where the phenomenon is known as Lamb shift [97]. In Section 4.1 we consider the special case that the degeneracy is lifted at second order in formal perturbation theory once an interaction is turned on. In Section 4.2 we study the situation where the degeneracy is caused by a symmetry. To keep notation simple we treat only the ground state. However we expect that resonances can be treated by the same ideas as used in the Sections 4.1 and 4.2 with additional notational complexity. We note that degenerate situations do occur in physically realistic models [6]. Moreover it is natural to assume that, if there exists a representation of the set of symmetries that acts reducible on an eigenspace of the Hamiltonian, then the degeneracy of the eigenstate is lifted by higher order perturbation theory up to the point where symmetries of the underlying system act irreducibly on the remaining eigenspace. In Section 4.3 we study a specific example of a quantum mechanical system that exhibits a degenerate ground state due to an intrinsic symmetry of the considered system. We use the result of Section 4.2 to show that the ground states and the ground-state eigenvalue are nevertheless real-analytic function of the coupling constant.

4.1 Second order split-up for the Spin-Boson model

In this section we extend operator-theoretic renormalization to situations where the unperturbed eigenvalue is degenerate and the degeneracy is lifted after the interaction is turned on. To keep notation simple we restrict our analysis to the ground state of the system.

More precisely, we consider a quantum mechanical atomic system described by a Hamilton operator acting on a Hilbert space. We call this Hilbert space the atomic space and we assume that it is finite-dimensional. We expect that this assumption is not essential and can be relaxed. We assume that the atomic system interacts with a quantized field of massless bosons by means of a linear coupling. Moreover, we assume that the interaction satisfies a mild infrared condition, which is needed for the renormalization analysis to converge. Furthermore, we assume that the Hamiltonian of the atomic system has a degenerate ground state, that is lifted by formal second order perturbation theory in the coupling constant. Note that first order perturbation theory does not affect the ground-state energy for the class of models we consider. In particular we use a generalized Spin-Boson Hamiltonian to describe the total energy of the coupled system. We show that the ground state of this operator exists for small values of the coupling constant, a result already known in the literature [60, 67, 99, 127]. Furthermore, we show that the ground-state projection as well as the ground-state energy are analytic as a function of the coupling constant in an open cone with apex at the origin. This result was published in [78] and it is in contrast to non-degenerate situations, where it has been shown that the ground-state projection and the ground-state energy are analytic functions of the coupling constant [66]. We do not assume that this is an artefact of our proof. In fact, we conjecture that in the degenerate case the ground-state projection and possibly the ground-state energy can be non-analytic in a neighborhood of zero. We note that non-analyticity in the fine-structure constant for a hydrogen atom that is minimally coupled to the quantized electromagnetic field was shown in [25].

Remark 4.1.1. Although we do not obtain analyticity in a neighborhood of zero, analyticity in a cone is of interest in its own right. It is for example a necessary ingredient to show Borel summability. Borel summability methods permit to recover a function from its asymptotic expansion. An asymptotic expansion is an approximation that is in some sense still good enough to recover, under specific conditions and using certain methods, the original function (cf. Remark 4.1.27). Such an asymptotic expansion may for example be obtained using the techniques employed in [12, 16, 17, 31, 69]. It is worth mentioning that we study the method presented in [31] in Chapter 5.

4.1.1 Definition of the model and statement of result

We consider the following model. Let the atomic Hilbert space be modeled by

$$\mathcal{H}_{\mathrm{at}} := \mathbb{C}^N$$

equipped with the standard scalar product. The Hilbert space of the total system is given by

$$\mathcal{H} := \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F} \,,$$

where $\mathcal{F}$ is the bosonic Fock space which was defined in Chapter 2. We use the notation given in Eq. (2.5) and denote again by $a^*(k)$ and $a(k)$ the operator-valued distributions satisfying canonical commutation relations. Note that these distributions are sometimes called the *usual creation and annihilation operators*. As in Chapter 2 we define the free field Hamiltonian by

$$H_f := \int \omega(k)\, a^*(k)\, a(k) dk \,, \tag{4.1}$$

where it is again given in the sense of forms. Moreover, we assume that $H_{\mathrm{at}} \in \mathcal{L}(\mathcal{H}_{\mathrm{at}})$ is self-adjoint. For $g \in \mathbb{C}$ we study the following Spin-Boson Hamiltonian

$$H_g := H_{\mathrm{at}} + H_f + gW \,, \tag{4.2}$$

where the interaction is given by

$$W = a^*(\omega^{-1/2} G) + a(\omega^{-1/2} G) \,. \tag{4.3}$$

We note that W is infinitesimally bounded with respect to H_f if $\omega^{-1}G$, $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$. For a formal definition of the smeared annihilation and creation operators we refer the reader to Section 3.3. Let ϵ_{at} denote the ground-state eigenvalue of the operator H_{at}, and let P_{at} denote the projection onto the eigenspace of H_{at} with eigenvalue ϵ_{at}. Moreover we set $\overline{P}_{\mathrm{at}} := \mathbb{1} - P_{\mathrm{at}}$ and define

$$Z_{\mathrm{at}} := -\int P_{\mathrm{at}}\, G^*(k) \left[\frac{P_{\mathrm{at}}}{|k|} + \frac{\overline{P}_{\mathrm{at}}}{H_{\mathrm{at}} - \epsilon_{\mathrm{at}} + |k|} \right] G(k)\, P_{\mathrm{at}} \frac{dk}{\omega(k)} \quad \restriction \mathrm{Ran} P_{\mathrm{at}} \,, \tag{4.4}$$

which is a self-adjoint mapping on the ground-state space of H_{at}. For $r > 0$ we denote the open disk in the complex plane by

$$D_r := \{ z \in \mathbb{C} : |z| < r \} \,.$$

Recall that $\mathcal{L}(\mathcal{H}_{\mathrm{at}})$ is equipped with the operator norm (cf. Definition 2.1.1), which we denote by $\|\cdot\|$ hereinafter. For the renormalization analysis to be applicable we need an infrared condition. Therefore we define for $\mu > 0$ the following space of measurable, operator-valued functions

$$L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}})) := \{ G : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{L}(\mathcal{H}_{\mathrm{at}}) : G \text{ measurable }, \|G\|_\mu < \infty \},$$

where we set

$$\|G\|_\mu := \int \left(\frac{1}{|k|^{3+2\mu}} + 1 \right) \|G(k)\|^2 dk \,. \tag{4.5}$$

Now we can state the main theorem of this section.

Theorem 4.1.2. *Let $\mu > 0$. Suppose $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\rm at}))$ and let H_g be given by Eq. (4.2). Let $\epsilon^{(2)}_{\rm at}$ denote the smallest eigenvalue of $Z_{\rm at}$ and assume that $\epsilon^{(2)}_{\rm at}$ is simple. Moreover let $0 < \delta_0 < \pi/2$, and let*

$$S_{\delta_0} := \{z \in \mathbb{C} : \ |\arg(z)| < \delta_0 \ \text{or} \ |\arg(-z)| < \delta_0\} .$$

Then there exists a $g_0 > 0$ such that for all $g \in D_{g_0} \cap S_{\delta_0}$ the operator H_g has an eigenvector ψ_g and an eigenvalue E_g such that

$$E_g = \epsilon_{\rm at} + g^2 \epsilon^{(2)}_{\rm at} + o(|g|^2) . \tag{4.6}$$

The eigenvalue and eigenprojection are continuous on $S_{\delta_0} \cap D_{g_0}$ and analytic in the interior of $S_{\delta_0} \cap D_{g_0}$. Furthermore for real g the number E_g is the infimum of the spectrum of H_g.

Remark 4.1.3. Let P_Ω denote the projection onto the vacuum vector Ω. Then we have

$$Z_{\rm at} \simeq -(P_{\rm at} \otimes P_\Omega) W (H_0 - \epsilon_{\rm at})^{-1} W (P_{\rm at} \otimes P_\Omega) \restriction {\rm Ran} P_{\rm at} \otimes P_\Omega , \tag{4.7}$$

which is exactly the second order energy correction in formal perturbation theory [115].

Remark 4.1.4. In Chapter 2 we defined the number operator $N := d\Gamma(\mathbb{1})$. As a quadratic form it can be represented by $N = \int a^*(k) a(k) dk$. Moreover we get the following relation $(-1)^N H_g (-1)^N = H_{-g}$. Therefore the eigenvalues of H_g do not depend on the sign of g. Moreover, if the eigenvalues happen to have an asymptotic expansion, this expansion cannot depend on odd powers of g.

The remaining parts of this section are devoted to the proof of Theorem 4.1.2. As was mentioned before the proof is based on operator-theoretic renormalization. For an overview on this method we refer the reader to Section 3.2. To prove the theorem we need to have control on the degeneracy. Therefore we perform two Feshbach projections before we initiate the renormalization procedure. The first Feshbach projection projects onto the spectral subspace of field energies between zero and ρ_0. With it we can control the resolvent in a neighborhood of the unperturbed ground-state energy. The second Feshbach projection projects onto the spectral subspace of even smaller field energies between zero and $\rho_0\rho_1$. This allows us to resolve the degeneracy. In the proof we choose ρ_0 larger than $|g|$ but $\rho_0\rho_1$ smaller than $|g|^2$. More precisely, we show the following.
For any $\rho_0 > 0$ there exists a $\rho_1 > 0$ and positive numbers $g_-(\rho_0)$, $g_+(\rho_0)$ with $g_-(\rho_0) < g_+(\rho_0)$, uniformly in the model parameters, such that for all coupling constants g in a sectorial region of the complex plane with

$$g_-(\rho_0) < |g| < g_+(\rho_0) \tag{4.8}$$

both Feshbach projections are isospectral and respect necessary Banach space estimates needed for operator-theoretic renormalization to be applicable. Such a sectorial region is depicted in Figure 4.1. By invoking an analyticity result of Griesemer and Hasler [66], we obtain analyticity of the ground state and ground-state energy for g in such a sectorial region. Moreover, we show that we can choose $g_-(\rho_0)$ such that $g_-(\rho_0) \to 0$ as $\rho_0 \to 0$. Note that at the same time $g_+(\rho_0) \to 0$. But this is no problem as long as $g_-(\rho_0) < g_+(\rho_0)$. Hence the analyticity in a cone with apex at the origin follows as ρ_0 tends to zero.

Remark 4.1.5. Degeneracies which are formally lifted at higher than second order should be treatable similarly by performing several initial Feshbach projections where the energy cutoffs as well depend on the coupling constant.

The remainder of this section is organized as follows. In Subsection 4.1.2 we define a first Feshbach projection with parameter ρ_0 and prove a Feshbach pair criterion, which establishes isospectrality. For more details on this we refer to Section 3.2. Then we make use of the Banach spaces of matrix-valued integral kernels introduced in Section 3.3. More precisely in Subsection 4.1.3 we show that the first Feshbach operator lies in a suitable neighborhood of the free field energy with respect to the norms introduced in Subsection 3.3.1. In Subsection 4.1.4 we define the second Feshbach projection with parameter ρ_1. We establish necessary estimates that we use in a later subsection to show the Feshbach pair criterion for the second step. We note that in order for these estimates, in particular Lemma 4.1.17, to hold we need that the coupling constant lies in a cone. We conclude the subsection with an abstract Feshbach pair criterion. In Subsection 4.1.5 we prove an abstract Banach space estimate for the second Feshbach step. We need this estimate later to show that the second Feshbach operator lies in a suitable neighborhood of the free field energy. Subsection 4.1.6 is devoted to the proof of the main result (Theorem 4.1.2). More precisely we use the abstract results of the previous two subsections and choose ρ_1 as a function of ρ_0. We make this choice such that the second Feshbach projection satisfies the abstract Feshbach pair criterion, establishing isospectrality, and such that the second Feshbach operator lies in a suitable neighborhood of the free field energy, allowing us to apply the analyticity result [66]. Furthermore, the situation illustrated in Figure 4.1 together with Eq. (4.8) is rigorously justified (cf. Eq. (4.82)).

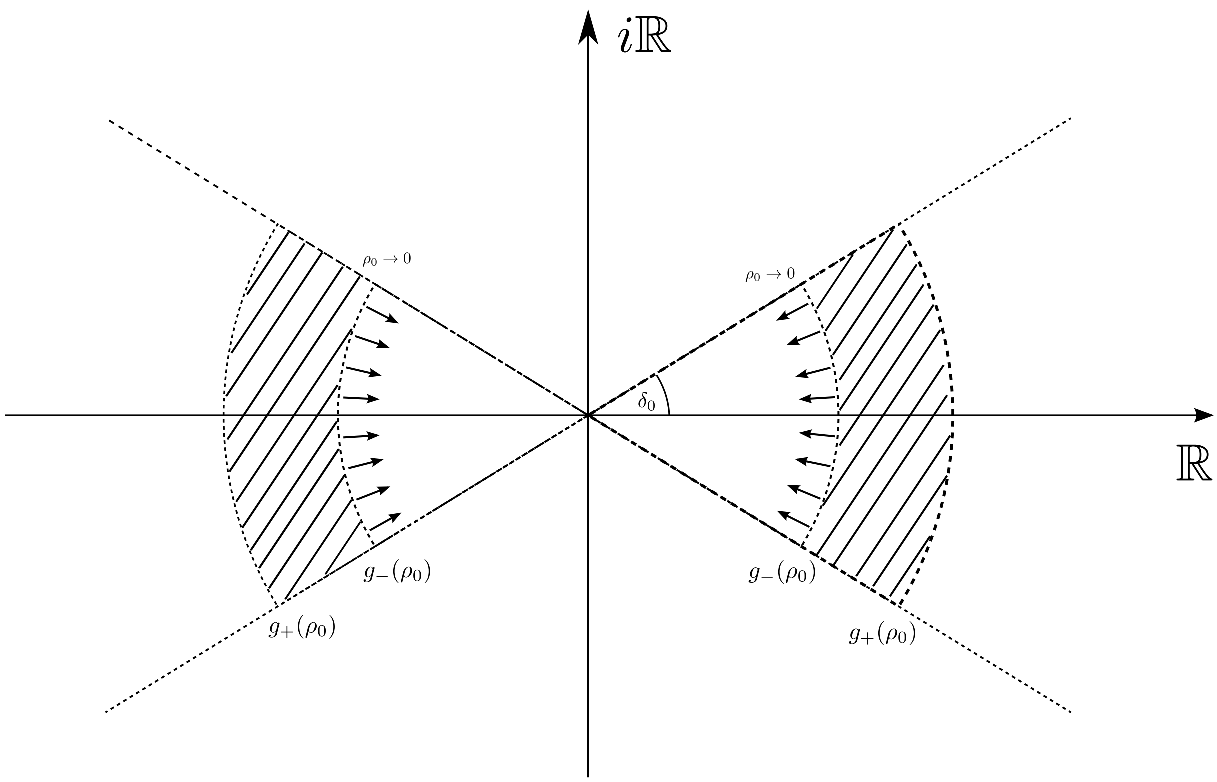


Figure 4.1: For fixed $\rho_0 > 0$, the ground state and the ground-state energy are analytic functions of the coupling constant in the shaded region.

4.1.2 Initial Feshbach step

In this subsection we define the initial Feshbach operator. Without loss of generality we assume that the distance of ϵ_{at} from the rest of the spectrum of H_{at} is 1, i.e.,

$$d_{\mathrm{at}} := \inf \sigma(H_{\mathrm{at}} \setminus \{\epsilon_{\mathrm{at}}\}) - \epsilon_{\mathrm{at}} = 1 . \tag{4.9}$$

This can always be achieved by a suitable scaling. We show that for z in a neighborhood of ϵ_{at} and for $|g|$ sufficiently small, the operators $H_g - z$ and $H_0 - z$ are a Feshbach pair for a generalized projection. We fix two functions χ and $\overline{\chi}$ in $C^\infty(\mathbb{R};[0,1])$ satisfying $\chi^2 + \overline{\chi}^2 = 1$ and

$$\chi(r) = \begin{cases} 1 , & \text{if } r \leq \frac{3}{4} , \\ 0 , & \text{if } r \geq 1 . \end{cases}$$

For $\rho > 0$ we define operator-valued functions

$$\boldsymbol{\chi}_\rho^{(0)}(r) := P_{\mathrm{at}} \otimes \chi(r/\rho) , \qquad \overline{\boldsymbol{\chi}}_\rho^{(0)}(r) := \overline{P}_{\mathrm{at}} \otimes \mathbb{1} + P_{\mathrm{at}} \otimes \overline{\chi}(r/\rho) ,$$

and by means of the spectral theorem we define the following linear operators on $\mathcal{H}$,

$$\boldsymbol{\chi}_\rho^{(0)} := \boldsymbol{\chi}_\rho^{(0)}(H_f) , \qquad \overline{\boldsymbol{\chi}}_\rho^{(0)} := \overline{\boldsymbol{\chi}}_\rho^{(0)}(H_f) .$$

It is easily verified that $\big(\boldsymbol{\chi}_\rho^{(0)}\big)^2 + \big(\overline{\boldsymbol{\chi}}_\rho^{(0)}\big)^2 = \mathbb{1}$.
The following theorem provides us with conditions for which we can define the initial Feshbach operator.

Theorem 4.1.6 (Feshbach pair criterion for 1st iteration).
Let $0 < \rho \leq 1/4$, $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$ and $\omega^{-1/2}G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$. The operators $H_g - z$ and $H_0 - z$ are a Feshbach pair for $\boldsymbol{\chi}_\rho^{(0)}$, if

$$|g| < \frac{\rho^{1/2}}{10\|\omega^{-1}G\|} . \tag{4.10}$$

Furthermore one has the absolutely convergent expansion

$$F_{\chi_{\rho_0}^{(0)}}(H_g - z, H_0 - z) = H_{\mathrm{at}} - z + H_f + \sum_{L=1}^{\infty} (-1)^{L-1} \chi_\rho^{(0)} gW \left(\frac{(\overline{\chi}_\rho^{(0)})^2}{H_0 - z} gW \right)^{L-1} \chi_\rho^{(0)}. \tag{4.11}$$

We now want to provide a proof of Theorem 4.1.6. For this we make use of the following two lemmas.

Lemma 4.1.7. *Let $\rho \geq 0$. Then for $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$ we have*

$$\left\| (H_f + \rho)^{-1/2} [a(G) + a^*(G)] (H_f + \rho)^{-1/2} \right\| \leq 2 \, \|\omega^{-1/2} G\| \, \rho^{-1/2}.$$

Proof. This assertion follows from Eq. (3.19) in Lemma 3.3.12. Especially we get for $r \geq 0$ the following estimate for the annihilation operator

$$\begin{aligned} \left\| (H_f + r)^{-1/2} a(G) (H_f + r)^{-1/2} \right\| &\leq \left\| (H_f + r)^{-1/2} \right\| \left\| a(G) H_f^{-1/2} \right\| \left\| H_f^{1/2} (H_f + r)^{-1/2} \right\| \\ &\leq r^{-1/2} \|\omega^{-1/2} G\| \,. \end{aligned}$$

Analogously we achieve the corresponding expression involving a creation operator. □

Lemma 4.1.8. *Let $0 < \rho \leq 1/4$ and $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$. The operator $H_0 - z$ is invertible on the range of $\overline{\chi}_\rho^{(0)}$ and we have the bound*

$$\left\| (H_0 - z)^{-1} \restriction \mathrm{Ran} \overline{\chi}_\rho^{(0)} \right\| \leq \frac{4}{\rho} \,, \tag{4.12}$$

and for all $\tau \geq 0$ the bound

$$\left\| (H_f + \tau)^{1/2} (H_0 - z)^{-1} (H_f + \tau)^{1/2} \restriction \mathrm{Ran} \overline{\chi}_\rho^{(0)} \right\| \leq 1 + \frac{4\tau}{\rho} \,. \tag{4.13}$$

Proof. We start by proving that $H_0 - z$ is bounded invertible on the range of $\overline{\chi}_\rho^{(0)}$. First we consider a normalized $\psi \in \mathrm{Ran}(P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho))$. It follows that

$$\|(H_0 - z)\psi\| \geq \inf{}_{r \geq \frac{3}{4}\rho} |\epsilon_{\mathrm{at}} + r - z| \geq (3/4 - 1/2)\rho = \frac{\rho}{4} \,.$$

Now we consider a normalized $\psi \in \mathrm{Ran}(\overline{P}_{\mathrm{at}} \otimes 1)$ and get that

$$\begin{aligned} \|(H_0 - z)\psi\| &\geq \inf{}_{r \geq 0} \left\| \big(H_{\mathrm{at}} \overline{P}_{\mathrm{at}} + r - z\big)\psi \right\| \\ &\geq \inf{}_{r \geq 0} \left\| \big(H_{\mathrm{at}} \overline{P}_{\mathrm{at}} - \epsilon_{\mathrm{at}} + r\big)\psi \right\| - |z - \epsilon_{\mathrm{at}}| \\ &\geq 1 - \rho/2 \,. \end{aligned}$$

Thus from the above two inequalities it follows that $H_0 - z$ is bounded invertible on the range of $\overline{\chi}_\rho^{(0)}$. Moreover we directly see that the bound (4.12) holds. In order to prove the second bound we use the following estimates

$$\begin{aligned} \|(H_f + \tau)^{1/2} (H_0 - z)^{-1} (H_f + \tau)^{1/2} \restriction \mathrm{Ran}(P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho))\| &= \sup_{r \geq \frac{3}{4}\rho} \left| \frac{r + \tau}{\epsilon_{\mathrm{at}} + r - z} \right| \\ &\leq 1 + \left| \frac{\tau}{(3/4 - 1/2)\rho} \right| \\ &\leq 1 + \frac{4\tau}{\rho} \,, \end{aligned}$$

and if we set $E_1 := \inf \big(\sigma(H_{\mathrm{at}}) \setminus \{\epsilon_{\mathrm{at}}\} \big)$

$$\begin{aligned} \left\| (H_f + \tau)^{1/2} (H_0 - z)^{-1} (H_f + \tau)^{1/2} \restriction \mathrm{Ran}(\overline{P}_{\mathrm{at}} \otimes 1) \right\| &= \sup_{r \geq 0} \sup_{\lambda \in \sigma(H_{\mathrm{at}}) \setminus \{\epsilon_{\mathrm{at}}\}} \left| \frac{r + \tau}{\lambda + r - z} \right| \\ &\leq \sup_{r \geq 0} \left| \frac{r + \tau}{E_1 - \epsilon_{\mathrm{at}} - \rho/2 + r} \right| \\ &\leq 1 + \frac{\tau}{1 - \rho/2} \,. \end{aligned}$$

These estimates confirm the bound (4.13). □

Proof of Theorem 4.1.6. We begin by verifying that H_0 and H_g are closed operators on the same domain. Since H_{at} is closed on $\mathcal{H}_{\mathrm{at}}$ and H_f is closed on $D(H_f)$ it follows that

$$H_0 = H_{\mathrm{at}} \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}_{\mathrm{at}}} \otimes H_f$$

is closed on $D(H_0) := \mathcal{H}_{\mathrm{at}} \otimes D(H_f)$. Therefore $H_g = H_0 + gW$ is closed on $D(H_0)$ provided the interaction operator W is infinitesimally bounded with respect to H_0. Using Lemma 4.1.8 and Lemma 4.1.7 this is easy to see. Since from the proof of Lemma 4.1.8 we deduce that $w := z - 1$ is in the resolvent set of H_0 for arbitrary $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$. Hence $H_f(H_0 - w)^{-1}$ is a bounded operator on $D(H_0)$. Now, using Eq. (4.13) and Lemma 4.1.7, we obtain for all $\varphi \in D(H_0)$ that W is infinitesimally bounded with respect to H_0. Thus H_g is closed on $D(H_0)$. Furthermore we additionally proved in Lemma 4.1.8 that H_0 is bounded invertible on $\mathrm{Ran}\overline{\boldsymbol{\chi}}_\rho^{(0)}$. Moreover we note that H_f and H_0 leave the range of $\overline{\boldsymbol{\chi}}_\rho^{(0)}$ invariant and that $(H_f + \tau)^{1/2}$ is bounded invertible on the range of $\overline{\boldsymbol{\chi}}_\rho^{(0)}$. In order to prove the bounded invertibility of $H_0 - z + g\overline{\boldsymbol{\chi}}_\rho^{(0)} W \overline{\boldsymbol{\chi}}_\rho^{(0)}$ we use again Lemmas 4.1.7 and 4.1.8. More precisely we define

$$\begin{aligned} A(z,\tau) &:= (H_f + \tau)^{-1/2}(H_0 - z)(H_f + \tau)^{-1/2}\,, \\ B(z,\tau,\rho) &:= (H_f + \tau)^{-1/2}\overline{\boldsymbol{\chi}}_\rho^{(0)} W \overline{\boldsymbol{\chi}}_\rho^{(0)} (H_f + \tau)^{-1/2}\,, \end{aligned}$$

and use the identity

$$\begin{aligned} &\big(H_0 - z + g\overline{\boldsymbol{\chi}}_\rho^{(0)} W \overline{\boldsymbol{\chi}}_\rho^{(0)}\big) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_\rho^{(0)} \\ &\qquad = (H_f + \tau)^{1/2}\big[A(z,\tau) + gB(z,\tau,\rho)\big](H_f + \tau)^{1/2} \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_\rho^{(0)} \\ &\qquad = (H_f + \tau)^{1/2} A(z,\tau)\big[1 + gA(z,\tau)^{-1}B(z,\tau,\rho)\big](H_f + \tau)^{1/2} \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_\rho^{(0)}\,. \end{aligned}$$

We see that the bounded invertibility follows from Neumann's Theorem (Theorem 3.2.9) provided

$$\|gA(z,\tau)^{-1}B(z,\tau,\rho)\| < 1\,.$$

Now applying Lemma 4.1.7, where we note that $\overline{\boldsymbol{\chi}}_\rho^{(0)}$ commutes with H_f, and using the bound (4.13) of Lemma 4.1.8, we obtain

$$\big\|gA(z,\tau)^{-1}B(z,\tau,\rho)\big\| \le |g| \left(1 + \frac{4\tau}{\rho}\right) 2\,\|\omega^{-1}G\|\,\tau^{-1/2}\,.$$

Thus, if we choose $\tau = \rho$, we have proved the theorem. □

4.1.3 Banach space estimate for the first step

In Subsection 4.1.2 we showed that the operators $H_g - z$, $H_0 - z$ are a Feshbach pair for $\boldsymbol{\chi}_\rho^{(0)}$, provided the coupling constant is in a sufficiently small neighborhood of zero and the spectral parameter is sufficiently close to the unperturbed ground-state energy. In particular, if the assumptions of Theorem 4.1.6 hold, we can define the operator

$$H_g^{(1,\rho)}(z) := S_\rho\big(F_{\boldsymbol{\chi}_\rho^{(0)}}(H_g - z, H_0 - z)\big), \tag{4.14}$$

which we call the *first Feshbach operator*. The goal of this subsection is to show that the first Feshbach operator is close to the free field energy. The distance is measured in terms of the norms introduced in the Section 3.3. More precisely, we define the following neighborhoods of the free field energy. For given $\alpha, \beta, \gamma \in \mathbb{R}_+$ we define $\mathcal{B}^{[d]}(\alpha,\beta,\gamma) \subset H(\mathcal{W}_\xi^{[d]})$ by

$$\mathcal{B}^{[d]}(\alpha,\beta,\gamma) := \Big\{H(w) : \|w_{0,0}(0)\| \le \alpha, \|w_{0,0}' - 1\|_\infty \le \beta, \|w - w_{0,0}\|_{\mu,\xi}^{\#} \le \gamma\Big\}\,, \tag{4.15}$$

where $w \in \mathcal{W}_\xi^{[d]}$ and we refer to Subsection 3.3.1 for detailed definitions of the occuring norms and spaces. We denote the dimension of the space of ground states of H_{at} by

$$d_0 := \dim(\mathrm{Ran}P_{\mathrm{at}})\,.$$

Moreover, we introduce the following global constants which we use in various estimates

$$C_F := 10\|\chi'\|_\infty + 20\,, \tag{4.16}$$

$$\hat{C}_F := 20\,. \tag{4.17}$$

For the renormalization analysis to be applicable, we need a stronger infrared condition. This condition is expressed in terms of the norm $\|\cdot\|_\mu$ defined in Eq. (4.5). As a direct consequence of that definition we have

$$\left\|\frac{G}{\omega}\right\| \le \|G\|_\mu\,. \tag{4.18}$$

This inequality shows that the criterion for the Feshbach pair property obtained in Theorem 4.1.6 can be expressed in terms of $\|G\|_\mu$. Now we can state the main theorem of this subsection.

Theorem 4.1.9 (Banach space estimate for 1st Feshbach operator)**.**
Let $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$, and $0 < \xi < 1$. Then there exist constants C_1, C_2, C_3, such that, if $0 < \rho < 1/4$, $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$ and

$$|g| < C_0 \rho^{1/2}\,, \qquad \textit{where} \quad C_0 := \frac{1}{8\,\xi^{-1}\,C_F\,\|G\|_\mu}\,, \tag{4.19}$$

the pair of operators $(H_g - z, H_0 - z)$ is a Feshbach pair for $\chi_\rho^{(0)}$, and

$$H_g^{(1,\rho)}(z) - \rho^{-1}(\epsilon_{\mathrm{at}} - z) \in \mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0)\,,$$

for

$$\alpha_0 = C_1\,|g|^2\rho^{-1}\,, \quad \beta_0 = C_2\,|g|^2\rho^{-1}\,, \quad \gamma_0 = C_3\,\rho^\mu(|g| + \rho^{-1}|g|^2 + \rho^{-2}|g|^3)\,.$$

Remark 4.1.10. The explicit form of the constants C_1, C_2, and C_3 can be read off from the Inequalities (4.41)–(4.43) which are obtained in the proof of the theorem. Note that only C_3 depends on ξ.

The remaining part of this subsection is devoted to the proof of Theorem 4.1.9. First observe that, in view of Ineq. (4.18) and Eq. (4.16), Assumption (4.19) implies the Assumption (4.10) of Theorem 4.1.6. Hence $(H_g - z, H_0 - z)$ is a Feshbach pair for $\chi_\rho^{(0)}$ provided g lies in a neighborhood of zero and z in a neighborhood of the unperturbed ground-state energy. Moreover we can expand the resolvent in the first Feshbach operator in an absolutely convergent Neumann series, cf. Eq. (4.11). Using the Pull-Through Formula (Lemma 3.3.11) and applying the generalized Wick's theorem (Theorem 3.3.14) we then put this series into normal order. But before we do that we first introduce an alternative notation for our original Hamiltonian. For this we define

$$\begin{aligned}
w^{(0)}_{g,0,0}(z)(r) &:= H_{\mathrm{at}} - z + r\,,\\
w^{(0)}_{g,1,0}(z)(r,k) &:= g\,G(k)\,,\\
w^{(0)}_{g,0,1}(z)(r,\tilde{k}) &:= g\,G^*(\tilde{k})\,.
\end{aligned} \tag{4.20}$$

We set $w_g^{(0)} := (w^{(0)}_{g,m,n})_{0\le m+n\le 1}$ and use the notation

$$H^{(0)}_{m,n}(w_{m,n}) := \int\limits_{A^{m+n}} a^*(k^{(m)})\; w_{m,n}(H_f, K^{(m,n)})\, a(\tilde{k}^{(n)})\,\frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}}\,. \tag{4.21}$$

We note that this expression is analogous to Eq. (3.10), apart from the fact that the domain of integration is different and there are no projections onto the reduced Fock space $\mathcal{H}_{\mathrm{red}}$. In order to highlight these differences we use a superscript zeroth order. In the new notation the interaction (4.3) reads

$$gW = H^{(0)}_{1,0}(w^{(0)}_{g,1,0}) + H^{(0)}_{0,1}(w^{(0)}_{g,0,1})\,. \tag{4.22}$$

For the bookkeeping of the terms in the Neumann expansion we introduce the following multi-indices for $L \in \mathbb{N}$,

$$\begin{aligned}
\underline{m} &:= (m_1, ..., m_L) \in \mathbb{N}_0^L\,,\\
|\underline{m}| &:= m_1 + \cdots + m_L\,,\\
\underline{0} &:= (0, ..., 0) \in \mathbb{N}_0^L\,.
\end{aligned}$$

Now inserting the alternative expression for the interaction, Eq. (4.22), into the convergent Neumann Series (4.11), and using the generalized Wick's theorem (Theorem 3.3.14) we obtain a sum of terms of the form

$$\begin{aligned} &V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w](r, K^{(|\underline{m}|,|\underline{n}|)}) \qquad (4.23)\\ &\quad := (P_{\rm at} \otimes P_\Omega) F_0^{(0,\rho)}[w](H_f + \rho(r + \tilde{r}_0))\\ &\qquad \prod_{l=1}^{L} \left\{ W^{(0)}{}^{m_l,n_l}_{p_l,q_l}[w](\rho K^{(m_l,n_l)}) F_l^{(0,\rho)}[w](H_f + \rho(r+\tilde{r}_l)) \right\} (P_{\rm at} \otimes P_\Omega) , \end{aligned}$$

where the definition of $\tilde{r}_l$ is given in Eq. (3.23), and where we used the following definitions

$$W^{(0)}{}^{m,n}_{p,q}[w](K^{(m,n)}) := \int\limits_{(\mathbb{R}^3\times\{1,2\})^{p+q}} a^*(x^{(p)})\, w_{m+p,n+q}[k^{(m)}, x^{(p)}, \tilde{k}^{(n)}, \tilde{x}^{(q)}]\, a(\tilde{x}^{(q)}) \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}} , \qquad (4.24)$$

$$\begin{aligned} F_l^{(0,\rho)}[w](r) &= F^{(0,\rho)}[w](r) := \frac{\left(\overline{\chi}_\rho^{(0)}\right)^2(r)}{w_{0,0}(r)} , \qquad \text{for } l = 1, ..., L-1 ,\\ F_0^{(0,\rho)}[w](r) &= F_L^{(0,\rho)}[w](r) := \chi(r/\rho) . \end{aligned}$$

We use the natural convention that there is no integration if $p = q = 0$ and that the argument $K^{(m,n)}$ is dropped if $m = n = 0$. Note that the appearance of the ρ's in the arguments on the right hand side of Eq. (4.23) is due to the scaling transformation S_ρ in Eq. (4.14). Thus we have shown the algebraic part of the following result.

Proposition 4.1.11. *Suppose the assumptions of Theorem 4.1.6 hold, i.e., let $0 < \rho \le \frac{1}{4}$, $z \in D_{\rho/2}(\epsilon_{\rm at})$ and Eq. (4.10) is satisfied. Define*

$$\hat{w}^{(1,\rho)}_{g,0,0}(z)(r) := \rho^{-1} \left(\epsilon_{\rm at} - z + \rho r + \sum_{L=2}^{\infty} (-1)^{L+1} \sum_{\underline{p},\underline{q} \in \mathbb{N}_0^L : p_l + q_l = 1} V^{(0,\rho)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[w_g^{(0)}(z)](r) \right) , \qquad (4.25)$$

and for $M, N \in \mathbb{N}_0$ with $M + N \ge 1$ define

$$\begin{aligned} &\hat{w}^{(1,\rho)}_{g,M,N}(z)(r, K^{(M,N)}) \qquad (4.26)\\ &\quad := \sum_{L=1}^{\infty} (-1)^{L+1} \rho^{M+N-1} \sum_{\substack{\underline{m},\underline{p},\underline{n},\underline{q} \in \mathbb{N}_0^L :\\ |\underline{m}| = M, |\underline{n}| = N,\\ m_l + p_l + q_l + n_l = 1}} V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w_g^{(0)}(z)](r, K^{(M,N)}) . \end{aligned}$$

Assume that the right hand sides converge with respect to the norm $\|\cdot\|^{\#}_{\mu,\xi}$ for some $\mu > 0$ and $\xi \in (0,1)$. Then for the symmetrization $w_g^{(1,\rho)}(z) := [\hat{w}_g^{(1,\rho)}(z)]^{\rm sym}$ we have

$$H_g^{(1,\rho)}(z) = H(w_g^{(1,\rho)}(z)) .$$

Proof. Let $\hat{w}_g^{(1,\rho,L_0)}$ be defined as the integral kernel obtained by the right hand sides of Eq. (4.25) and (4.26), if we sum L only up to L_0. Then by the absolute convergence of the Neumann Series (4.11) and an application of the generalized Wick's theorem (Theorem 3.3.14) as discussed above, we find

$$H_g^{(1,\rho)}(z) = \lim_{L_0\to\infty} H(\hat{w}_g^{(1,\rho,L_0)}) .$$

A detailed description of how to obtain the integral kernels is given in [66, Appendix A], we refer also to [14, Theorem 3.7]. The assumption that the right hand sides of Eq. (4.25) and (4.26) converge with respect to the norm $\|\cdot\|^{\#}_{\mu,\xi}$ implies in view of Theorem 3.3.4 that

$$\begin{aligned} \lim_{L_0\to\infty} H(\hat{w}_g^{(1,\rho,L_0)}) &= \lim_{L_0\to\infty} H([\hat{w}_g^{(1,\rho,L_0)}]^{\rm sym})\\ &= H(\lim_{L_0\to\infty} [\hat{w}_g^{(1,\rho,L_0)}]^{\rm sym}) = H(w_g^{(1,\rho)}(z)) . \end{aligned}$$

□

Our next goal is to show Inequalities (4.41), (4.42), and (4.43), below. These estimates imply on the one hand that the right hand sides of Eq. (4.25) and (4.26) converge with respect to the norm $\|\cdot\|^{\#}_{\mu,\xi}$, and on the other hand establish a proof of Theorem 4.1.9. To obtain the desired estimates we use the bounds collected in the subsequent proposition.

Proposition 4.1.12. *For all $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$, $0 < \rho < 1/4$, $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$, $L \in \mathbb{N}$, and $\underline{m}, \underline{p}, \underline{n}, \underline{q} \in \mathbb{N}_0^L$ we have*

$$\begin{aligned} &\rho^{|\underline{m}|+|\underline{n}|-1} \big\| V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g] \big\|^{\#}_{\mu} \\ &\quad \le (L+2)\hat{C}_F^{L-1}(1+\|\chi'\|_\infty)|g|^L \rho^{-L+\frac{1}{2}(|\underline{p}|-p_1+|\underline{q}|-q_L)} \rho^{(1+\mu)(|\underline{m}|+|\underline{n}|)} \left\| \frac{G}{\omega} \right\|^{|\underline{p}|+|\underline{q}|} \|G\|_\mu^{|\underline{m}|+|\underline{n}|}, \end{aligned} \tag{4.27}$$

and

$$\rho^{-1} \big\| V^{(0,\rho)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[w^{(0)}_g] \big\|_\infty \le \hat{C}_F^{L-1} |g|^L \rho^{-L+\frac{1}{2}(|\underline{p}|+|\underline{q}|)} \left\| \frac{G}{\omega} \right\|^{|\underline{p}|+|\underline{q}|}, \tag{4.28}$$

$$\rho^{-1} \big\| \partial_r V^{(0,\rho)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[w^{(0)}_g] \big\|_\infty \le (L+1)\hat{C}_F^{L-1}(1+\|\chi'\|_\infty) |g|^L \rho^{-L+\frac{1}{2}(|\underline{p}|+|\underline{q}|)} \left\| \frac{G}{\omega} \right\|^{|\underline{p}|+|\underline{q}|}, \tag{4.29}$$

where $\underline{0} \in \mathbb{N}_0^L$.

Remark 4.1.13. We note that in contrast to Eq. (4.67) in Lemma 4.1.23 (see also [14, Lemma 3.10]) we have an additional factor $\rho^{\frac{1}{2}(|\underline{p}|-p_1+|\underline{q}|-q_L)}$, which yields an improved estimate. The proof of Theorem 4.1.9, which we present, makes use of this improved estimate.

In order to show the above proposition we need the estimates from the following lemma.

Lemma 4.1.14. *For $\rho \ge 0$ we set $\Xi_\rho := H_f + \rho$.*

(a) Let $\omega^{-1/2} G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$. Then for all $m, n, p, q \in \mathbb{N}_0$, with $m+n+p+q = 1$, all $K^{(m,n)} \in B_1^{m+n}$, and $\rho \ge 0$ we have

$$\big\| \Xi_\rho^{-1/2} \, {W^{(0)}}^{m,n}_{p,q}[w^{(0)}_g](K^{(m,n)}) \, \Xi_\rho^{-1/2} \big\| \le \left\| \frac{G}{\omega} \right\|^{p+q} |g| \{\|G(k_1)\|\}^m \{\|G(\tilde{k}_1)\|\}^n \rho^{\frac{1}{2}(p+q)-1}, \tag{4.30}$$

$$\big\| \mathbb{1}_{[0,1]}(H_f) \, {W^{(0)}}^{m,n}_{p,q}[w^{(0)}_g](K^{(m,n)}) \, \Xi_\rho^{-1/2} \big\| \le \left\| \frac{G}{\omega} \right\|^{p+q} |g| \, \{\|G(k_1)\|\}^m \{\|G(\tilde{k}_1)\|\}^n \rho^{\frac{1}{2}(q-1)}, \tag{4.31}$$

$$\big\| \Xi_\rho^{-1/2} \, {W^{(0)}}^{m,n}_{p,q}[w^{(0)}_g](K^{(m,n)}) \, \mathbb{1}_{[0,1]}(H_f) \big\| \le \left\| \frac{G}{\omega} \right\|^{p+q} |g| \, \{\|G(k_1)\|\}^m \{\|G(\tilde{k}_1)\|\}^n \rho^{\frac{1}{2}(p-1)}. \tag{4.32}$$

(b) For all $0 < \rho < 1/4$, $z \in D_{\rho/2}(\epsilon_{\mathrm{at}})$ and $r \in [0, \infty)$ we have

$$\big\| \Xi_\rho^{1/2} \, F^{(0,\rho)}[w^{(0)}_g(z)](H_f + \rho r) \, \Xi_\rho^{1/2} \big\| \le 5 \le \hat{C}_F, \tag{4.33}$$

$$\big\| \Xi_\rho^{1/2} \, \partial_r F^{(0,\rho)}[w^{(0)}_g(z)](H_f + r\rho) \, \Xi_\rho^{1/2} \big\| \le 20 + 10 \, \|\chi'\|_\infty \le \hat{C}_F \, (1 + \|\chi'\|_\infty). \tag{4.34}$$

Proof. Equation (4.30) follows directly from the proof of Lemma 4.1.7. In addition, Eqns. (4.31) and (4.32) follow from the proof of Lemma 4.1.7 with help of Eq. (3.20). Estimate (4.33) follows from Lemma 4.1.8. And in order to show Estimate (4.34) we calculate the derivative

$$\partial_r F^{(0,\rho)}[w^{(0)}_g](H_f + r\rho) = \frac{2\, \overline{\chi}^{(0)}_\rho(H_f + r\rho)\, (\overline{\chi}^{(0)}_1)'(\rho^{-1} H_f + r)}{H_{\mathrm{at}} + H_f + r\rho - z} + \frac{\big(\overline{\chi}^{(0)}_\rho\big)^2(H_f + r\rho)\, \rho}{(H_{\mathrm{at}} + H_f + r\rho - z)^2}. \tag{4.35}$$

In order to estimate the terms on the right hand side of Eq. (4.35) we use again Lemma 4.1.8, together with

$$\left\| \frac{\Xi_\rho^{1/2}}{\Xi_{\rho+r\rho}^{1/2}} \right\| \le 1. \tag{4.36}$$

This concludes the proof of Lemma 4.1.14. □

Proof of Proposition 4.1.12. Let $\Xi_\rho := H_f + \rho$. We estimate $\|V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g]\|_\mu$ using

$$\big|\langle \varphi_{\mathrm{at}} \otimes \Omega,\ A_1 A_2 \cdots A_n\, \varphi_{\mathrm{at}} \otimes \Omega\rangle\big| \le \|A_1\|\|A_2\| \cdots \|A_n\|\,, \tag{4.37}$$

where $\|\cdot\|$ denotes the operator norm, and Inequalities (4.30)–(4.34). We get for $r \ge 0$,

$$\begin{aligned}
&\big\|V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g](r, K^{(|\underline{m}|,|\underline{n}|)})\big\| \\
&\le \Big\|(P_{\mathrm{at}} \otimes P_\Omega)\, F^{(0,\rho)}_0[w^{(0)}_g](H_f + \rho(r+\tilde r_0))\, W^{(0)}{}^{m_1,n_1}_{p_1,q_1}[w^{(0)}_g](\rho K^{(m_1,n_1)})\, \Xi_\rho^{-1/2} \\
&\qquad \times \Xi_\rho^{1/2}\, F^{(0,\rho)}_1[w^{(0)}_g](H_f + \rho(r+\tilde r_1))\, \Xi_\rho^{1/2} \\
&\qquad \times \prod_{l=2}^{L-1} \Big\{ \Xi_\rho^{-1/2}\, W^{(0)}{}^{m_l,n_l}_{p_l,q_l}[w^{(0)}_g](\rho K^{(m_l,n_l)})\, \Xi_\rho^{-1/2}\, \Xi_\rho^{1/2}\, F^{(0,\rho)}_l[w^{(0)}_g](H_f + \rho(r+\tilde r_l))\, \Xi_\rho^{1/2} \Big\} \\
&\qquad \times \Xi_\rho^{-1/2}\, W^{(0)}{}^{m_L,n_L}_{p_L,q_L}[w^{(0)}_g](\rho K^{(m_l,n_l)})\, F^{(0,\rho)}_L[w^{(0)}_g](H_f + \rho(r+\tilde r_L))\, (P_{\mathrm{at}} \otimes P_\Omega)\Big\| \\
&\le \hat C_F^{L-1} \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} |g|^L \Big[\prod_{l=1}^{L} [\|G(\rho k_{m_l})\|]^{m_l} [\|G(\rho \tilde k_{n_l})\|]^{n_l}\Big] \Big[\prod_{l=2}^{L-1} \rho^{\frac12(p_l+q_l)-1}\Big] \rho^{\frac12(q_1+p_L)-1} \\
&\le \hat C_F^{L-1} \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} |g|^L \rho^{-L+1} \rho^{\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \prod_{l=1}^{L} [\|G(\rho k_{m_l})\|]^{m_l} [\|G(\rho \tilde k_{n_l})\|]^{n_l}\,.
\end{aligned} \tag{4.38}$$

In order to compute the derivative we use Leibniz' rule. Applying it and using Estimate (4.37) and the Inequalities (4.30)–(4.34) we obtain for $r \ge 0$

$$\begin{aligned}
&\big\|\partial_r V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g](r, K^{(|\underline{m}|,|\underline{n}|)})\big\| \\
&\le \hat C_F^{L-1}(1 + \|\chi'\|_\infty)\,(L+1) \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} |g|^L \rho^{-L+1+\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \prod_{l=1}^{L} [\|G(\rho k_{m_l})\|]^{m_l} [\|G(\rho \tilde k_{n_l})\|]^{n_l}\,.
\end{aligned} \tag{4.39}$$

Now we can estimate, inserting (4.38) and (4.39), respectively,

$$\begin{aligned}
\big\|V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g]\big\|_\mu &= \bigg(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \big\|V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g](K^{(|\underline{m}|,|\underline{n}|)})\big\|^2_\infty \frac{dK^{(\underline{m},\underline{n})}}{|K^{(\underline{m},\underline{n})}|^{3+2\mu}} \bigg)^{1/2} \\
&\le \hat C_F^{L-1}\, |g|^L \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} \rho^{-L+1+\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \\
&\qquad \times \bigg(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \prod_{l=1}^{L} \Big\{ \|G(\rho k_{m_l})\|^{2m_l} \|G(\rho \tilde k_{n_l})\|]^{2n_l} \Big\} \frac{dK^{(\underline{m},\underline{n})}}{|K^{(\underline{m},\underline{n})}|^{3+2\mu}} \bigg)^{1/2} \\
&\le \hat C_F^{L-1}\, |g|^L \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} \rho^{-L+1+\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \rho^{\mu(|\underline{m}|+|\underline{n}|)} \|G\|_\mu^{|\underline{m}|+|\underline{n}|}\,, \\
\big\|\partial_r V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g]\big\|_\mu &= \bigg(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \big\|\partial_r V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g](K^{(|\underline{m}|,|\underline{n}|)})\big\|^2_\infty \frac{dK^{(\underline{m},\underline{n})}}{|K^{(\underline{m},\underline{n})}|^{3+2\mu}} \bigg)^{1/2} \\
&\le (L+1)\, \hat C_F^{L-1}(1 + \|\chi'\|_\infty)\, |g|^L \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} \rho^{-L+1+\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \\
&\qquad \times \bigg(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \prod_{l=1}^{L} \Big\{ \|G(\rho k_{m_l})\|^{2m_l} \|G(\rho \tilde k_{n_l})\|]^{2n_l} \Big\} \frac{dK^{(\underline{m},\underline{n})}}{|K^{(\underline{m},\underline{n})}|^{3+2\mu}} \bigg)^{1/2} \\
&\le (L+1)\, \hat C_F^{L-1}(1 + \|\chi'\|_\infty) |g|^L \Big\|\frac{G}{\omega}\Big\|^{|\underline{p}|+|\underline{q}|} \rho^{-L+1+\frac12(|\underline{p}|-p_1+|\underline{q}|-q_L)} \rho^{\mu(|\underline{m}|+|\underline{n}|)} \|G\|_\mu^{|\underline{m}|+|\underline{n}|}.
\end{aligned}$$

Adding above estimates yields Eq. (4.27). Eqns. (4.28) and (4.29) follow similarly noting that $|\underline{m}| = 0$ and $|\underline{n}| = 0$ can only occur if L is even and on the very left we have an annihilation operator and on the very right a creation operator. $\square$

Proof of Theorem 4.1.9. It suffices to establish Inequalities (4.41)–(4.43) below. Let $S^L_{M,N}$ denote the set of tuples $(\underline{m}, \underline{p}, \underline{n}, \underline{q}) \in \mathbb{N}_0^{4L}$ with $|\underline{m}| = M$, $|\underline{n}| = N$, and

$$m_l + p_l + q_l + n_l = 1\,. \tag{4.40}$$

Such tuples obviously satisfy $|\underline{m}| + |\underline{n}| + |\underline{n}| + |\underline{q}| = L$. Using this identity, we now estimate the norm of Eq. (4.26) using Eqns. (4.27) and (4.18). This yields

$$\begin{aligned}
\big\|(w^{(1,\rho)}_{g,M,N})_{M+N\geq 1}(z)\big\|^{\#}_{\mu,\xi} &= \sum_{M+N\geq 1} \xi^{-(M+N)} \big\|w^{(1,\rho)}_{g,M,N}(z)\big\|^{\#}_{\mu} \\
&\leq \sum_{M+N\geq 1} \sum_{L=1}^{\infty} \sum_{(\underline{m},\underline{p},\underline{n},\underline{q})\in S^L_{M,N}} \xi^{-(M+N)} \rho^{M+N-1} \big\|V^{(0,\rho)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[w^{(0)}_g(z)]\big\|^{\#}_{\mu} \\
&\leq \sum_{L=1}^{\infty} \sum_{M+N\geq 1} (1 + \|\chi'\|_\infty)\,(L+2)\,|g|^L\, \hat{C}_F^{L-1} \|G\|^L_\mu \\
&\qquad \sum_{(\underline{m},\underline{p},\underline{n},\underline{q})\in S^L_{M,N}} \xi^{-|\underline{m}|-|\underline{n}|} \rho^{-L/2+(\frac{1}{2}+\mu)(|\underline{m}|+|\underline{n}|)-\frac{1}{2}(p_1+q_L)} \\
&\leq \rho^\mu (1 + \|\chi'\|_\infty) \Big[\rho^{1/2}\, 6\, \xi^{-1}|g|\,\rho^{-1/2} \|G\|_\mu \\
&\qquad + 64 \left(\xi^{-1}|g|\,\rho^{-1/2}\, \hat{C}_F\, \|G\|_\mu\right)^2 \\
&\qquad + \rho^{-1/2} \sum_{L=3}^{\infty} (L+2) \left(4\,\xi^{-1}|g|\,\rho^{-1/2}\, \hat{C}_F\, \|G\|_\mu\right)^L \Big]\,,
\end{aligned} \tag{4.41}$$

where in the last inequality we estimated the summands with $L = 1$ and $L = 2$ separately and summed over the terms with $L \geq 3$, as we now explain. First we note that Eq. (4.40) implies that $S^L_{M,N}$ is empty unless $M + N \leq L$, and that the number of elements $(\underline{m}, \underline{p}, \underline{n}, \underline{q}) \in \mathbb{N}_0^{4L}$ which satisfy (4.40) is bounded above by 4^L. Specifically for $L = 1$, we have only two terms: (m_1, p_1, n_1, q_1) equal $(1, 0, 0, 0)$ or equal $(0, 0, 1, 0)$. For $L = 2$ we use that $1 \leq M + N$ implies $p_1 + q_L \leq 1$. For $L \geq 3$ we use $|\underline{m}| + |\underline{n}| - (p_1 + q_L) \geq -1$. These considerations establish Estimate (4.41). Now we estimate the norm of Eq. (4.25) using Estimate (4.29), by means of a similar but simpler estimate

$$\begin{aligned}
\sup_{r\in[0,1]} \big|\partial_r w^{(1,\rho)}_{g,0,0}(z)(r) - 1\big| &\leq \rho^{-1} \sum_{L=2}^{\infty} \sum_{\substack{\underline{p},\underline{q}\in\mathbb{N}_0^L:\\ p_l+q_l=1}} \big\|\partial_r V^{(0,\rho)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[w^{(0)}_g(z)]\big\|_\infty \\
&\leq (1 + \|\chi'\|_\infty) \sum_{L=2}^{\infty} (L+1) \left(2 \left\|\frac{G}{\omega}\right\| |g|\,\rho^{-1/2}\, \hat{C}_F\right)^L .
\end{aligned} \tag{4.42}$$

Using Estimate (4.28) we analogously obtain

$$\begin{aligned}
\big|w^{(1,\rho)}_{g,0,0}(z)(0) + \rho^{-1}(z - \epsilon_{\mathrm{at}})\big| &\leq \rho^{-1} \sum_{L=2}^{\infty} \sum_{\substack{\underline{p},\underline{q}\in\mathbb{N}_0^L:\\ p_l+q_l=1}} \big\|V^{(0,\rho)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[w^{(0)}_g(z)]\big\|_\infty \\
&\leq \sum_{L=2}^{\infty} \left(2 \left\|\frac{G}{\omega}\right\| |g|\,\rho^{-1/2}\, \hat{C}_F\right)^L .
\end{aligned} \tag{4.43}$$

The series on the right hand sides of the Inequalities (4.41)–(4.43) converge if Eq. (4.19) holds. Hence Theorem 4.1.9 follows in view of these converging inequalities. □

4.1.4 Second Feshbach step

In this subsection we perform our second Feshbach step. Henceforth we denote by ρ_0 the field energy cutoff of the first Feshbach step and by ρ_1 the field energy cutoff of the second Feshbach step. We begin by approximating $w^{(1,\rho_0)}_{g,0,0}(z)$, which is the content of Lemma 4.1.15. Then we prove that this approximation

is invertible on the range of a suitable projection operator (Lemma 4.1.17). At the end of this subsection we show an abstract Feshbach pair criterion for the second step (Theorem 4.1.19).
To begin with we recall the mapping Z_{at}, which was defined in Eq. (4.4). Moreover recall that its ground-state eigenvalue, $\epsilon_{\mathrm{at}}^{(2)}$, is by assumption a simple eigenvalue.

Lemma 4.1.15 (Free approximation to 1st Feshbach operator)**.**
There exists a constant C such that the following holds. Let $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$, $0 < \rho_0 < 1/4$, and suppose

$$|g| < \frac{\rho_0^{1/2}}{4\, C_F\, \|\omega^{-1} G\|}\,. \tag{4.44}$$

If $z \in D_{\rho_0/2}(\epsilon_{\mathrm{at}})$, then the defining series of $w_{g,0,0}^{(1,\rho_0)}(z)$, i.e., the right hand side of Eq. (4.25), converges absolutely and

$$\sup_{0 \le r \le 1} \left\| \rho_0^{-1}\big(\epsilon_{\mathrm{at}} - z + \rho_0 r + \chi^2(r)\, g^2 Z_{\mathrm{at}}\big) - w_{g,0,0}^{(1,\rho_0)}(z)(r) \right\| \le C\left(\|G\|_\mu\, |g|^2 + \|G\|_\mu^4\, C_F^4\, |g|^4\, \rho_0^{-2} \right).$$

Remark 4.1.16. If we replace the $\chi^2(r)$ in front of Z_{at} by one, then the proof given below would yield a bound only of order $|g|^2\, \rho_0^{-1}$.

Proof. From Eq. (4.25) we see that

$$w_{g,0,0}^{(1,\rho_0)}(z)(r) = \rho_0^{-1} \left(\epsilon_{\mathrm{at}} - z + \rho_0 r + \sum_{L=2}^{\infty} (-1)^{L+1} \sum_{\underline{p},\underline{q} \in \mathbb{N}_0^L : p_l + q_l = 1} V^{(0,\rho_0)}_{(\underline{0},\underline{p},\underline{0},\underline{q})}[w_g^{(0)}(z)](r) \right).$$

The summand with $L = 2$ is only non-vanishing if $\underline{p} = (0, 1)$ and $\underline{q} = (1, 0)$. Moreover summands with odd L vanish. These claims follow directly from Eq. (4.23). Using this we can write

$$\begin{aligned}
&\rho_0^{-1}\big(\epsilon_{\mathrm{at}} - z + \rho_0 r + \chi^2(r)\, g^2 Z_{\mathrm{at}}\big) - w_{g,0,0}^{(1,\rho_0)}(z)(r) \\
&\qquad = \rho_0^{-1}\Big(\chi^2(r)\, g^2 Z_{\mathrm{at}} - X_g^{\rho_0}(z)(r) \Big) - Y_g^{\rho_0}(z)(r) \\
&\qquad = \rho_0^{-1}\Big(\chi^2(r)\, g^2 Z_{\mathrm{at}} - X_g^{\rho_0}(\epsilon_{\mathrm{at}})(r) + X_g^{\rho_0}(\epsilon_{\mathrm{at}})(r) - X_g^{\rho_0}(z)(r) \Big) - Y_g^{\rho_0}(z)(r)\,,
\end{aligned} \tag{4.45}$$

where we introduced the notation

$$\begin{aligned}
X_g^{\rho_0}(z)(r) &:= -V^{(0,\rho_0)}_{(\underline{0},(0,1),\underline{0},(1,0))}[w_g^{(0)}(z)](r)\,, \\
Y_g^{\rho_0}(z)(r) &:= \rho_0^{-1} \sum_{L=4}^{\infty} (-1)^{L+1} \sum_{\underline{p},\underline{q} \in \mathbb{N}_0^L : p_l + q_l = 1} V^{(0,\rho_0)}_{(\underline{0},\underline{p},\underline{0},\underline{q})}[w_g^{(0)}(z)](r)\,.
\end{aligned}$$

The second term can be estimated similarly to Eq. (4.43), i.e., using Eq. (4.28) we find

$$\sup_{r \in [0,1]} \|Y_g^{\rho_0}(z)(r)\| \le \sum_{L=4}^{\infty} \left(2\,|g| \left\| \frac{G}{\omega} \right\| \hat{C}_F\, \rho_0^{-1/2} \right)^L. \tag{4.46}$$

In order to obtain a suitable estimate for $X_g^{\rho_0}$ we use that $(\overline{\boldsymbol{\chi}}_{\rho_0}^{(0)})^2$ has a natural decomposition into a sum of two terms and we calculate the vacuum expectation using the Pull-Through Formula (Lemma 3.3.11)

$$\begin{aligned}
X_g^{\rho_0}(z)(r) = {}& g^2 \chi(r)\, P_{\mathrm{at}} \int \left\{ G^*(k)\, P_{\mathrm{at}}\, \frac{\overline{\chi}^2(\rho_0^{-1}|k| + r)}{\epsilon_{\mathrm{at}} - z + |k| + \rho_0 r}\, P_{\mathrm{at}}\, G(k) \right\} \frac{dk}{|k|}\, P_{\mathrm{at}}\, \chi(r) \\
&+ g^2 \chi(r)\, P_{\mathrm{at}} \int \left\{ G^*(k)\, \overline{P}_{\mathrm{at}}\, \frac{1}{H_{\mathrm{at}} - z + |k| + \rho_0 r} \overline{P}_{\mathrm{at}}\, G(k) \right\} \frac{dk}{|k|}\, P_{\mathrm{at}}\, \chi(r)\,.
\end{aligned} \tag{4.47}$$

First we estimate the relative error if z is replaced by ϵ_{at}. That is we show for $z \in D_{\rho_0/2}(\epsilon_{\mathrm{at}})$,

$$\sup_{r \in [0,1]} \left\| X_g^{\rho_0}(z)(r) - X_g^{\rho_0}(\epsilon_{\mathrm{at}})(r) \right\| \le |g|^2 \rho_0\, (3 + 1)\, \|G\|_\mu\,. \tag{4.48}$$

To estimate the first term in Eq. (4.47) we use common denominators

$$\begin{aligned}\frac{1}{\epsilon_{\rm at}-z+|k|+\rho_0 r}-\frac{1}{|k|+\rho_0 r} &= \frac{1}{\epsilon_{\rm at}-z+|k|+\rho_0 r}\,(z-\epsilon_{\rm at})\,\frac{1}{|k|+\rho_0 r}\\ &= (z-\epsilon_{\rm at})\left[1+\frac{z-\epsilon_{\rm at}}{\epsilon_{\rm at}-z+|k|+\rho_0 r}\right]\frac{1}{(|k|+\rho_0 r)^2}\,. \qquad (4.49)\end{aligned}$$

This yields

$$\overline{\chi}^2(\rho_0^{-1}|k|+r)\,\big|\,\text{l.h.s. of (4.49)}\,\big| \le \rho_0\left(1+\frac{\rho_0}{\frac{3}{4}\rho_0-\frac{1}{4}\rho_0}\right)|k|^{-2}\,,$$

where l.h.s. means 'left hand side'. This estimate explains the first contribution on the right hand side of Eq. (4.48). We estimate the second term in Eq. (4.47) similarly. For $E\in\sigma(H_{\rm at})\setminus\{\epsilon_{\rm at}\}$ we write

$$\begin{aligned}&\frac{1}{E-z+|k|+\rho_0 r}-\frac{1}{E-\epsilon_{\rm at}+|k|+\rho_0 r} \qquad (4.50)\\ &= \frac{1}{E-z+|k|+\rho_0 r}(z-\epsilon_{\rm at})\frac{1}{E-\epsilon_{\rm at}+|k|+\rho_0 r}\,,\end{aligned}$$

This yields

$$\big|\,\text{l.h.s. of (4.50)}\,\big| \le \rho_0\,|k|^{-2}\,.$$

This explains the second contribution on the right hand side of Eq. (4.48). Next we show that

$$\sup_{r\in[0,1]}\left\|\chi^2(r)g^2 Z_{\rm at}-X_g^{\rho_0}(\epsilon_{\rm at})(r)\right\| \le 3\,\|G\|_\mu\,|g|^2\,\rho_0\,. \qquad (4.51)$$

To estimate the first term in Eq. (4.47) with $z=\epsilon_{\rm at}$ we use

$$\frac{1}{|k|+\rho_0 r}-\frac{1}{|k|}=\frac{1}{|k|+\rho_0 r}(-\rho_0 r)\frac{1}{|k|}\,,$$

and make use of

$$|\overline{\chi}^2(\rho_0^{-1}|k|+r)-1| \le \begin{cases} 0\,, & |k|\ge\rho_0\,,\\ 1\,, & |k|\le\rho_0\,.\end{cases}$$

We estimate the second term in Eq. (4.47) with $z=\epsilon_{\rm at}$ using for $E\in\sigma(H_{\rm at})\setminus\{\epsilon_{\rm at}\}$ that

$$\frac{1}{E-\epsilon_{\rm at}+|k|+\rho_0 r}-\frac{1}{E-\epsilon_{\rm at}+|k|}=\frac{1}{E-\epsilon_{\rm at}+|k|+\rho_0 r}(-\rho_0 r)\frac{1}{E-\epsilon_{\rm at}+|k|}\,.$$

This gives Eq. (4.51). Finally inserting Estimates (4.46), (4.48) and (4.51) into Eq. (4.45) finishes the proof of the lemma. □

Let $P_{\rm at}^{(2)}$ denote the projection onto the one-dimensional eigenspace of $Z_{\rm at}$ with eigenvalue $\epsilon_{\rm at}^{(2)}$ and let $\overline{P}_{\rm at}^{(2)}=\mathbb{1}-P_{\rm at}^{(2)}$. We note that the superscript (2) originates from the fact that these expressions are obtained by formal second order perturbation theory. For $\rho_1>0$ we define

$$\begin{aligned}\boldsymbol{\chi}_{\rho_1}^{(1)}(r) &= P_{\rm at}^{(2)}\otimes\chi(r/\rho_1)\,,\\ \overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}(r) &= \overline{P}_{\rm at}^{(2)}\otimes\mathbb{1}+P_{\rm at}^{(2)}\otimes\overline{\chi}(r/\rho_1)\,,\end{aligned}$$

and

$$\boldsymbol{\chi}_{\rho_1}^{(1)}=\boldsymbol{\chi}_{\rho_1}^{(1)}(H_f)\,,\qquad \overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}=\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}(H_f)\,.$$

In the following we denote by $d_{\rm at}^{(2)}$ the distance between the lowest and second lowest eigenvalue of $Z_{\rm at}$. By the assumption $0\le\delta_0<\pi/2$ the following expression is positive

$$c_{\delta_0} := \inf_{g\in S_{\delta_0}}|d_{\rm at}^{(2)}+g^{-2}|>0\,. \qquad (4.52)$$

This claim follows from an easy minimization problem, yielding

$$c_{\delta_0}=\begin{cases} d_{\rm at}^{(2)}\,, & \text{if } 0\le\delta_0\le\pi/4\,,\\ d_{\rm at}^{(2)}\sin(\pi-2\delta_0)\,, & \text{if } \pi/4<\delta_0<\pi/2\,.\end{cases}$$

Moreover, for $\rho_0 > 0$ we assume that the following two inequalities hold.

$$\rho_0^{-1}|g|^2 < \frac{\frac{1}{4}}{\|Z_{\mathrm{at}}\| + c_{\delta_0}}\,, \tag{4.53}$$

and

$$\rho_1\rho_0 \le |g|^2 c_{\delta_0}\,. \tag{4.54}$$

Now we can establishes the required invertibility of the free approximation from Lemma 4.1.15.

Lemma 4.1.17 (Invertibility of free approximation to 1st Feshbach operator)**.**
Suppose $\rho_0, \rho_1 \in (0, 1/2]$. *Let* $g \in S_{\delta_0}$ *satisfy Eq.* (4.53) *and* (4.54). *Then for* $z \in D_{\rho_0\rho_1/2}(\epsilon_{\mathrm{at}} + g^2\epsilon_{\mathrm{at}}^{(2)})$ *we have*

$$\left\| \Big(\rho_0^{-1}\big(\epsilon_{\mathrm{at}} - z + \rho_0 H_f + \chi^2(H_f) g^2 Z_{\mathrm{at}}\big) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)} \Big)^{-1} \right\| \le \frac{4}{\rho_1}\,. \tag{4.55}$$

Proof. For notational simplicity we shall write

$$X(\rho_0, g, z) := \rho_0^{-1}(\epsilon_{\mathrm{at}} - z + \rho_0 H_f + \chi^2(H_f) g^2 Z_{\mathrm{at}})\,.$$

Let ψ be normalized and in the range of $Q_1 := \overline{P}_{\mathrm{at}}^{(2)} \otimes \mathbb{1}_{[0,1]}(H_f)$, then we have

$$\begin{aligned} \big\|X(\rho_0, g, z)\psi\big\| &\ge \inf_{\eta \in \mathrm{Ran}\overline{P}_{\mathrm{at}}^{(2)},\, \|\eta\|=1} \ \inf_{0 \le r \le 1} \big\|\big(- \rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \rho_0^{-1}\chi^2(r) g^2 Z_{\mathrm{at}}\big)\eta\big\| - \big|\rho_0^{-1}\big(\epsilon_{\mathrm{at}} + g^2\epsilon_{\mathrm{at}}^{(2)} - z\big)\big| \\ &\ge \rho_0^{-1}|g|^2\, c_{\delta_0} - \frac{\rho_1}{2}\,, \end{aligned} \tag{4.56}$$

where we used in the last inequality on the one hand that for $r \in [0, 3/4]$ we have by Eq. (4.52),

$$\begin{aligned} \big\|(-\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \rho_0^{-1}\chi^2(r) g^2 Z_{\mathrm{at}})\eta\big\| &= \rho_0^{-1}|g|^2\, \big\|\big(- \epsilon_{\mathrm{at}}^{(2)} + g^{-2}\rho_0 r + Z_{\mathrm{at}}\big)\eta\big\| \\ &\ge \rho_0^{-1}|g|^2\, c_{\delta_0}, \end{aligned}$$

and on the other hand that we have for $r \in [3/4, 1]$,

$$\big\|(-\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \rho_0^{-1}\chi^2(r) g^2 Z_{\mathrm{at}})\eta\big\| \ge \frac{3}{4} - \rho_0^{-1}|g|^2\, \big\|Z_{\mathrm{at}} - \epsilon_{\mathrm{at}}^{(2)}\big\| \ge \rho_0^{-1}|g|^2\, c_{\delta_0}\,,$$

by Eq. (4.53). Using Inequality (4.54) in Eq. (4.56) it now follows that

$$\big\|(X(\rho_0, g, z) \restriction \mathrm{Ran}Q_1)^{-1}\big\| \le \frac{2}{\rho_1}\,.$$

Next we consider a normalized ψ in the range of $Q_2 := P_{\mathrm{at}}^{(2)} \otimes \overline{\chi}(H_f/\rho_1)$. We get

$$\begin{aligned} \big\|X(\rho_0, g, z)\psi\big\| &\ge \inf_{\substack{\eta \in \mathrm{Ran}P_{\mathrm{at}}^{(2)}, \|\eta\|=1,\\ 3\rho_1/4 \le r \le 1}} \big\|(-\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \chi^2(r)\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)})\eta\big\| - \big|\rho_0^{-1}(\epsilon_{\mathrm{at}} + g^2\epsilon_{\mathrm{at}}^{(2)} - z)\big| \\ &\ge \frac{3}{4}\rho_1 - \frac{1}{2}\rho_1 = \frac{1}{4}\rho_1\,, \end{aligned}$$

where we used that for $r \in [3\rho_1/4, 3/4]$ we have

$$\big\|(-\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \chi^2(r)\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)})\eta\big\| = r \ge \frac{3}{4}\rho_1\,,$$

and for $r \in [3/4, 1]$ we have

$$\big\|(-\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)} + r + \chi^2(r)\rho_0^{-1} g^2 \epsilon_{\mathrm{at}}^{(2)})\eta\big\| \ge \frac{3}{4} - \rho_0^{-1}|g|^2\, |\epsilon_{\mathrm{at}}^{(2)}| \ge \frac{1}{2} \ge \frac{3}{4}\,\rho_1\,,$$

by Eq. (4.53). Thus we can invert the operator $X(\rho_0, g, z)$ on the range of $\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}$. $\square$

In order to perform a second Feshbach iteration, we need a suitable decomposition of the first Feshbach operator (4.14). We already know that $H_g^{(1,\rho)} = H(w_g^{(1,\rho)})$ for a suitable series of integral kernels $w_g^{(1,\rho)}$ and $0 < \rho \leq 1/4$ (cf. Proposition 4.1.11). We define

$$t_g^{(1,\rho_0)} := P_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)} P_{\mathrm{at}}^{(2)} + \overline{P}_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)} \overline{P}_{\mathrm{at}}^{(2)} ,$$
$$w_{g,\mathrm{int}}^{(1,\rho_0)} := \big(P_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)} \overline{P}_{\mathrm{at}}^{(2)} + \overline{P}_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)} P_{\mathrm{at}}^{(2)} , w_{g,m+n\geq 1}^{(1,\rho_0)}\big) ,$$

and choose the decomposition

$$H_g^{(1,\rho_0)} = T_g^{(1,\rho_0)} + W_g^{(1,\rho_0)} , \tag{4.57}$$

where

$$T_g^{(1,\rho_0)} := H_{0,0}(t_g^{(1,\rho_0)}) , \tag{4.58}$$
$$W_g^{(1,\rho_0)} := H(w_{g,\mathrm{int}}^{(1,\rho_0)}) . \tag{4.59}$$

Remark 4.1.18. We note that decomposition (4.57) into free part and interacting part is not unique. The isospectrality property of the smooth Feshbach merely requires that the free part commutes with the smoothed projections. This issue is pointed out in [14, Remark 2.4]. A different possibility would be to use a decomposition according to

$$t_{g,\mathrm{free}}^{(1,\rho_0)}(z)(r) := \rho_0^{-1}(\epsilon_{\mathrm{at}} - z + \rho_0 r + \chi^2(r) g^2 Z_{\mathrm{at}}) , \tag{4.60}$$
$$w_{g,\mathrm{rest}}^{(1,\rho_0)} := \big(w_{g,0,0}^{(1,\rho_0)} - t_{g,\mathrm{free}}^{(1,\rho_0)} , w_{g,m+n\geq 1}^{(1,\rho_0)}\big) .$$

The proof of Theorem 4.1.2, given in Subsection 4.1.6, would carry through also with this decomposition, with only notational modifications.

Now we state and prove an abstract Feshbach pair criterion.

Theorem 4.1.19 (Abstract Feshbach pair criterion for 2nd iteration)**.**
Assume that the smallest eigenvalue of Z_{at} is simple. Let $\rho_1 \in (0,1]$. Suppose

$$t \in C\big([0,1]; \mathcal{L}(P_{\mathrm{at}}^{(2)} \mathcal{H}_{\mathrm{at}}) \oplus \mathcal{L}(\overline{P}_{\mathrm{at}}^{(2)} P_{\mathrm{at}} \mathcal{H}_{\mathrm{at}})\big) ,$$

and $w \in \mathcal{W}_\xi^{[d]}$. Then the operators $H_{0,0}(t)$ and $H(w)$ are a Feshbach pair for $\boldsymbol{\chi}_{\rho_1}^{(1)}$, provided

(i) $H_{0,0}(t)$ is invertible on the closure of $\mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}$ *and* $\|(H_{0,0}(t) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)})^{-1}\| \leq \frac{8}{\rho_1}$,

(ii) and

$$\|H(w)\| < \frac{\rho_1}{8} . \tag{4.61}$$

In this case we have the absolutely convergent expansion

$$F_{\boldsymbol{\chi}_{\rho_1}^{(1)}}(H_{0,0}(t), H(w)) = H_{0,0}(t) + \sum_{L=1}^{\infty} (-1)^{L-1} \boldsymbol{\chi}_{\rho_1}^{(1)} H(w) \left(\frac{(\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)})^2}{H_{0,0}(t)} H(w) \right)^{L-1} \boldsymbol{\chi}_{\rho_1}^{(1)} . \tag{4.62}$$

Proof. First observe that $H_{0,0}(t)$ commutes with $\boldsymbol{\chi}_{\rho_1}^{(1)}$. The Feshbach pair property follows from (i) and (ii) and Neumann's theorem (Theorem 3.2.9). The second claim follows again by Neumann's theorem. □

Remark 4.1.20. During the proof of the main theorem in Subsection 4.1.6 we determine an explicit relation among ρ_0 and ρ_1 and g. Using this relation we verify Assumption (i) and (ii) of Theorem 4.1.19 with the help of Lemma 4.1.15 and 4.1.17.

4.1.5 Banach Space estimate for the second step

In Subsection 4.1.6 we show that $(T_g^{(1,\rho_0)}(z), W_g^{(1,\rho_0)}(z))$ is indeed a Feshbach pair for $\boldsymbol{\chi}_{\rho_1}^{(1)}$. In order to do that we use estimates that we proved in the last subsection. Hence it is justified to assume for the moment that the Feshbach property is satisfied. In that case we can define the *second Feshbach operator*

$$H_g^{(2,\rho_1)}(z) := S_{\rho_1}(F_{\boldsymbol{\chi}_{\rho_1}^{(1)}}(T_g^{(1,\rho_0)}(z), W_g^{(1,\rho_0)}(z))) ,$$

provided the right sides exist. Similar to Subsection 4.1.3 we now aim to show that there exists a sequence of integrals kernels $w_g^{(2,\rho_1)}(z)$ such that

$$H(w_g^{(2,\rho_1)}(z)) = H_g^{(2,\rho_1)}(z) \restriction \mathrm{Ran}\boldsymbol{\chi}_{\rho_1}^{(1)} .$$

This assertion follows as a conclusion of the subsequent theorem.
For notational compactness we introduce in the current subsection the constant

$$C_\chi := 20\sqrt{2} .$$

Theorem 4.1.21 (Abstract Banach space estimate for 2nd Feshbach operator)**.**
Let $0 < \xi \leq 1/4$ and assume that the smallest eigenvalue of Z_{at} is simple. Moreover let $0 < \rho_1 \leq 1$ and suppose $w \in \mathcal{W}_\xi^{[d]}$,

$$t \in C^1\big([0,1]; \mathcal{L}(P_{\mathrm{at}}^{(2)}\mathcal{H}_{\mathrm{at}}) \oplus \mathcal{L}(\overline{P}_{\mathrm{at}}^{(2)} P_{\mathrm{at}}\mathcal{H}_{\mathrm{at}})\big) ,$$

and

(i) $H_{0,0}(t)$ is invertible on the closure of $\mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}$ and $\|(H_{0,0}(t) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)})^{-1}\| \leq \frac{8}{\rho_1}$,

(ii) $\|H(w)\| < \frac{\rho_1}{8}$.

Then $H(t)$ and $H(w)$ are a Feshbach pair for $\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}$. Moreover, suppose

$$\gamma < \frac{\rho_1}{8C_\chi} ,$$

and

$$\begin{aligned}
&\|w\|_{\mu,\xi}^{\#} \leq \gamma , \\
&\|t'\|_\infty \leq \tau_0 , \\
&\|P_{\mathrm{at}}^{(2)} t' P_{\mathrm{at}}^{(2)} - \mathbb{1}\| \leq \tau_1 .
\end{aligned}$$

Then

$$S_{\rho_1}\big(F_{\boldsymbol{\chi}_{\rho_1}^{(1)}}(H(t), H(w))\big) - \rho_1^{-1} P_{\mathrm{at}}^{(2)} t(0) P_{\mathrm{at}}^{(2)} \in \mathcal{B}^{[1]}(\alpha_1, \beta_1, \gamma_1) ,$$

where

$$\begin{aligned}
\alpha_1 &= 12\,\big(1 + 2\,\|\chi'\|_\infty + 8\tau_0\big)\, C_\chi\, \gamma\, \rho_1^{-1} , \\
\beta_1 &= \tau_1 + 12\,\big(1 + 2\,\|\chi'\|_\infty + 8\tau_0\big)\, C_\chi\, \gamma\, \rho_1^{-1} , \\
\gamma_1 &= 96\,\big(1 + 2\,\|\chi'\|_\infty + 8\tau_0\big)\, \rho_1^{\mu}\, C_\chi\, \gamma .
\end{aligned}$$

In order to prove Theorem 4.1.21 we first use the Neumann expansion given in Eq. (4.62), then we put the resulting expression in normal order using the generalized Wick's Theorem (Theorem 3.3.14). We summarize the result in Proposition 4.1.22 below. To state the proposition we introduce the following notation. Let $L \in \mathbb{N}$, then for $l = 0, L$ we define the expressions $F_l^{(1,\rho_1)}[t](r) := \chi(r/\rho_1)$ and for $l = 1, ..., L-1$ we set

$$F_l^{(1,\rho_1)}[t](r) = F^{(1,\rho_1)}[t](r) := \frac{\Big(\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}(r)\Big)^2}{t(r)} .$$

Moreover we define for $w \in \mathcal{W}_{m+p,n+q}^{[d]}$ the expression

$$\begin{aligned}
&W_{p,q}^{m,n}[w](r, K^{(m,n)}) \qquad\qquad (4.63)\\
&:= \mathbb{1}_{[0,1]}(H_f) \int_{B_1{}^{p+q}} a^*(x^{(p)})\, w_{m+p,n+q}[H_f + r, k^{(m)}, x^{(p)}, \tilde{k}^{(n)}, \tilde{x}^{(q)}]\, a(\tilde{x}^{(q)})\, \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}}\, \mathbb{1}_{[0,1]}(H_f) ,
\end{aligned}$$

where we use the natural convention that there is no integration if $p = q = 0$ and that the argument $K^{(m,n)}$ is dropped if $m = n = 0$. Furthermore we recall the notation

$$\underline{m} := (m_1, ..., m_L) \in \mathbb{N}_0^L , \qquad |\underline{m}| := m_1 + \cdots + m_L , \qquad \underline{0} := (0, ..., 0) \in \mathbb{N}_0^L .$$

With these expressions at hand we define

$$V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](r,K^{(|\underline{m}|,|\underline{n}|)})$$
$$:= (P^{(2)}_{\mathrm{at}} \otimes P_\Omega)\, F^{(1,\rho_1)}_0[t](H_f + \rho_1(r+\tilde r_0)) \tag{4.64}$$
$$\prod_{l=1}^{L} \Big\{ W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde r_l), \rho_1 K^{(m_l,n_l)})\, F^{(1,\rho_1)}_l[t](H_f + \rho_1(r+\tilde r_l)) \Big\}\, (P^{(2)}_{\mathrm{at}} \otimes P_\Omega)\,,$$

where $\tilde r_l$ is defined as in Eq. (3.23). We note that the notation is similar to the one in Subsection 4.1.3.

Proposition 4.1.22. *Suppose the assumptions of Theorem 4.1.21 hold. Define*

$$\hat w^{(2,\rho_1)}_{0,0}(r) := \rho_1^{-1}\left(t(\rho_1 r) + \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\underline{p},\underline{q}\in\mathbb{N}_0^L} V^{(1,\rho_1)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[t,w](r) \right), \tag{4.65}$$

and, for $M+N \ge 1$,

$$\hat w^{(2,\rho_1)}_{M,N}(r,K^{(M,N)}) \tag{4.66}$$
$$:= \sum_{L=1}^{\infty} (-1)^{L+1} \rho_1^{M+N-1} \sum_{\substack{\underline{m},\underline{p},\underline{n},\underline{q}\in\mathbb{N}_0^L \\ |\underline{m}|=M, |\underline{n}|=N}} \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \right\} V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](r,K^{(M,N)})\,.$$

Moreover assume that the right hand sides converge with respect to the norm $\|\cdot\|^{\#}_{\mu,\xi}$. Let $w^{(2,\rho_1)}$ be the symmetrization with respect to $k^{(M)}$ and $\tilde k^{(N)}$ of $\hat w^{(2,\rho_1)}$. Then

$$S_{\rho_1}\big(F_{\boldsymbol{\chi}^{(1)}_{\rho_1}}(H(t),H(w))\big) = H(w^{(2,\rho_1)})\,.$$

The proof of this proposition is analogous to the proof of Proposition 4.1.11 and we refer there for more details. In order to prove Theorem 4.1.21, we need the estimate of the following lemma.

Lemma 4.1.23. *Suppose the assumptions of Theorem 4.1.21 hold. For fixed $L\in\mathbb{N}$ and $\underline{m},\underline{p},\underline{n},\underline{q}\in\mathbb{N}_0^L$ and $w\in\mathcal{W}^{[d]}_\xi$ we have*

$$\rho_1^{(|\underline{m}|+|\underline{n}|)-1} \big\| V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w] \big\|^{\#}_{\mu} \tag{4.67}$$
$$\le (L+2)\, 2^{L/2}\, \hat C_\chi^{L-1} \left(1 + \|\chi'\|_\infty + 4\,\|t'\|_\infty\right) \rho_1^{(1+\mu)(|\underline{m}|+|\underline{n}|)-L} \prod_{l=1}^{L} \frac{\|w_{m_l+p_l,n_l+q_l}\|^{\#}_{\mu}}{\sqrt{p_l^{p_l} q_l^{q_l}}}\,.$$

with $\hat C_\chi = 20$ and the convention that $p^p := 1$ for $p=0$.

Remark 4.1.24. In contrast to [14] we do not have the conditions $m_l+p_l+q_l+n_l \ge 1$ and $p_l+q_l\ge 1$ in Eq. (4.66) and (4.65), respectively. The proof of Lemma 4.1.23 is still similar to the proof of Lemma 3.10 in [14], however we have to take into account more terms. Fortunately, these terms are hidden in the notation we introduced for our Banach spaces of matrix-valued integral kernels in Section 3.3.

Proof of Lemma 4.1.23. We start by estimating the resolvents. Let $0\le u+\rho_1 r\le 1$ for $u,r\ge 0$. Then for $l=0$ and $l=L$ we have

$$\big|F^{(1,\rho_1)}_l[t](u+\rho_1 r)\big| \le 1\,, \qquad \big|\partial_r F^{(1,\rho_1)}_l[t](u+\rho_1 r)\big| \le \|\chi'\|_\infty\,,$$

and for $l=1,...,L-1$,

$$\big\|F^{(1,\rho_1)}_l[t](u+\rho_1 r)\big\| \le \left\| \frac{\left(\overline{\boldsymbol{\chi}}^{(1)}_{\rho_1}(u+\rho_1 r)\right)^2}{t(u+\rho_1 r)} \right\| \le \frac{8}{\rho_1} \le \frac{\widehat C_\chi}{\rho_1}\,,$$

as well as

$$
\begin{aligned}
\left\|\partial_r F_l^{(1,\rho_1)}[t](u+\rho_1 r)\right\| &\leq \left\|\frac{2\overline{\chi}_{\rho_1}^{(1)}(u+\rho_1 r)\,\partial_r\overline{\chi}_{\rho_1}^{(1)}(u+\rho_1 r)}{t(u+\rho_1 r)}\right\| + \left\|\frac{\left(\overline{\chi}_{\rho_1}^{(1)}(u+\rho_1 r)\right)^2 \rho_1\, t'(u+\rho_1 r)}{(t(u+\rho_1 r))^2}\right\| \\
&\leq \frac{16}{\rho_1}\,\|\chi'\|_\infty + \frac{64}{\rho_1}\,\|t'\|_\infty \\
&\leq \frac{\widehat{C}_\chi}{\rho_1}\left(\|\chi'\|_\infty + 4\,\|t'\|_\infty\right),
\end{aligned}
$$

where we used an equation similar to (4.35). Now we estimate $\left\|V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w]\right\|$ and $\left\|\partial_r V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w]\right\|$ using a variant of Eq. (4.37).

$$
\begin{aligned}
\left\|V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](r,K^{(|\underline{m}|,|\underline{n}|)})\right\| &\leq \prod_{l=0}^{L}\left\|F_l^{(1,\rho_1)}[t](H_f+\rho_1(r+\tilde{r}_l))\right\| \prod_{l=1}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\| \\
&\leq \widehat{C}_\chi^{L-1}\,\rho_1^{-L+1}\prod_{l=1}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|. \qquad (4.68)
\end{aligned}
$$

Similarly we get with Leibniz' rule

$$
\begin{aligned}
&\left\|\partial_r V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](r,K^{(|\underline{m}|,|\underline{n}|)})\right\| \\
&\leq \left\{\sum_{j=0}^{L}\left\|\partial_r F_j^{(1,\rho_1)}[t](H_f+\rho_1(r+\tilde{r}_l))\right\| \prod_{\substack{l=0\\ l\neq j}}^{L}\left\|F_l^{(1,\rho_1)}[t](H_f+\rho_1(r+\tilde{r}_l))\right\|\right\} \\
&\qquad \prod_{l=1}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\| \\
&\quad + \prod_{l=0}^{L}\left\|F_l^{(1,\rho_1)}[t](H_f+\rho_1(r+\tilde{r}_l))\right\| \left\{\sum_{j=1}^{L}\left\|\partial_r W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|\right. \\
&\qquad \left.\prod_{\substack{l=1\\ l\neq j}}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|\right\} \\
&\leq (L+1)\,\widehat{C}_\chi^{L-1}\left(\|\chi'\|_\infty + 4\,\|t'\|_\infty\right)\rho_1^{-L+1}\prod_{l=1}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\| \\
&\quad + \widehat{C}_\chi^{L-1}\rho_1^{-L+1}\left\{\rho_1\sum_{j=1}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[\partial_r w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|\right. \\
&\qquad \left.\prod_{\substack{l=1\\ l\neq j}}^{L}\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|\right\} \\
&\leq (L+1)\,\widehat{C}_\chi^{L-1}\left(1+\|\chi'\|_\infty + 4\,\|t'\|_\infty\right)\rho_1^{-L+1} \\
&\quad \prod_{l=1}^{L}\left\{\left\|W^{m_l,n_l}_{p_l,q_l}[w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\| + \rho_1\left\|W^{m_l,n_l}_{p_l,q_l}[\partial_r w](\rho_1(r+\tilde{r}_l),\rho_1 K^{(m_l,n_l)})\right\|\right\}. \qquad (4.69)
\end{aligned}
$$

To estimate the $\|\cdot\|_\mu$-norm we use the following estimate

$$
\begin{aligned}
&\int_{B_1^{m_l+n_l}}\sup_{r\in[0,1]}\left\|W^{m_l,n_l}_{p_l,q_l}[w](r,\rho_1 K^{(m_l,n_l)})\right\|^2\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}} \\
&\qquad = \rho_1^{2\mu(m_l+n_l)}\int_{B_{\rho_1}^{m_l+n_l}}\sup_{r\in[0,1]}\left\|W^{m_l,n_l}_{p_l,q_l}[w](r,K^{(m_l,n_l)})\right\|^2\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}} \\
&\qquad \leq \rho_1^{2\mu(m_l+n_l)}\,\frac{1}{p_l^{p_l}\,q_l^{q_l}}\int_{B_{\rho_1}^{m_l+n_l}}\|w_{m_l+p_l,n_l+q_l}(\cdot\,,\,\cdot\,,K^{(m_l,n_l)})\|_\mu^2\,\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}} \\
&\qquad \leq \rho_1^{2\mu(m_l+n_l)}\,\frac{1}{p_l^{p_l}\,q_l^{q_l}}\,\|w_{m_l+p_l,n_l+q_l}\|_\mu^2, \qquad (4.70)
\end{aligned}
$$

where the first equality follows by the substitution formula for integrals, the second line follows from the estimate in Lemma 3.3.2, and the last line follows from Fubini's theorem. Using Estimate (4.68) together with (4.70) we find

$$\begin{aligned}\big\|V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w]\big\|_\mu &= \left(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \big\|V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](K^{(|\underline{m}|,|\underline{n}|)})\big\|_\infty^2 \frac{dK^{(|\underline{m}|,|\underline{n}|)}}{|K^{(|\underline{m}|,|\underline{n}|)}|^{3+2\mu}}\right)^{1/2}\\
&\le \widehat{C}_\chi^{L-1}\,\rho_1^{-L+1}\prod_{l=1}^L\left\{\int_{B_1^{m_l+n_l}}\sup_{r\in[0,1]}\big\|W^{m_l,n_l}_{p_l,q_l}[w](r,\rho_1K^{(m_l,n_l)})\big\|^2\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}}\right\}^{1/2}\\
&\le \widehat{C}_\chi^{L-1}\,\rho_1^{-L+1}\,\rho_1^{\mu(|\underline{m}|+|\underline{n}|)}\prod_{l=1}^L\left\{\frac{1}{\sqrt{p_l^{p_l}q_l^{q_l}}}\,\|w_{m_l+p_l,n_l+q_l}\|_\mu\right\}. \qquad (4.71)\end{aligned}$$

Similarly using Estimate (4.69) together with (4.70) we find

$$\begin{aligned}&\big\|\partial_r V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](K^{(|\underline{m}|,|\underline{n}|)})\big\|_\mu\\
&= \left(\int_{B_1^{|\underline{m}|+|\underline{n}|}} \big\|\partial_r V^{(1,\rho_1)}_{\underline{m},\underline{p},\underline{n},\underline{q}}[t,w](K^{(|\underline{m}|,|\underline{n}|)})\big\|_\infty^2 \frac{dK^{(|\underline{m}|,|\underline{n}|)}}{|K^{(|\underline{m}|,|\underline{n}|)}|^{3+2\mu}}\right)^{1/2}\\
&\le (L+1)\,\widehat{C}_\chi^{L-1}\left(1+\|\chi'\|_\infty+4\,\|t'\|_\infty\right)\rho_1^{-L+1}\\
&\quad\prod_{l=1}^L\Bigg\{\int_{B_1^{m_l+n_l}}\sup_{r\in[0,1]}\Big\{\big\|W^{m_l,n_l}_{p_l,q_l}[w](r,\rho_1K^{(m_l,n_l)})\big\|+\rho_1\big\|W^{m_l,n_l}_{p_l,q_l}[\partial_r w](r,\rho_1K^{(m_l,n_l)})\big\|\Big\}^2\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}}\Bigg\}^{1/2}\\
&\le (L+1)\,\widehat{C}_\chi^{L-1}\left(1+\|\chi'\|_\infty+4\,\|t'\|_\infty\right)\rho_1^{-L+1}\\
&\quad\prod_{l=1}^L\Bigg\{2\int_{B_1^{m_l+n_l}}\Big(\sup_{r\in[0,1]}\big\|W^{m_l,n_l}_{p_l,q_l}[w](r,\rho_1K^{(m_l,n_l)})\big\|^2\\
&\qquad\qquad+\rho_1\sup_{r\in[0,1]}\big\|W^{m_l,n_l}_{p_l,q_l}[\partial_r w](r,\rho_1K^{(m_l,n_l)})\big\|^2\Big)\frac{dK^{(m_l,n_l)}}{|K^{(m_l,n_l)}|^{3+2\mu}}\Bigg\}^{1/2}\\
&\le (L+1)\,\widehat{C}_\chi^{L-1}\left(1+\|\chi'\|_\infty+4\,\|t'\|_\infty\right)\rho_1^{-L+1}\rho_1^{\mu(|\underline{m}|+|\underline{n}|)}\prod_{l=1}^L\left\{\frac{\sqrt{2}}{\sqrt{p_l^{p_l}q_l^{q_l}}}\|w_{m_l+p_l,n_l+q_l}\|_\mu^\#\right\}. \qquad (4.72)\end{aligned}$$

Adding the Estimates (4.71) and (4.72) establishes the desired Inequality (4.67). □

In the following we prove Theorem 4.1.21. We note that the presented proof is similar to the proof of Theorem 3.8 in [14].

Proof of Theorem 4.1.21. Due to Theorem 4.1.19 the operators $H(t)$ and $H(w)$ are a Feshbach pair for $\overline{\chi}^{(1)}_{\rho_1}$. We begin this proof by establishing an estimate for Eq. (4.66). To do this we use Lemma 4.1.23 and set for notational simplicity $C_t := 1+2\|\chi'\|_\infty+8\tau_0$. Since $C_\chi := \sqrt{2}\,\hat{C}_\chi$ and $\hat{C}_\chi\ge 1$, we have

$$2^{L/2}\,\hat{C}_\chi^{L-1}\le C_\chi^L\,.$$

Moreover, we use that $\binom{m+p}{p}\le 2^{m+p}$. Thus we find for $M+N\ge 1$

$$\begin{aligned}\|w^{(2,\rho_1)}_{M,N}\|_\mu^\# &\le \sum_{L=1}^\infty\sum_{\substack{\underline{m},\underline{p},\underline{n},\underline{q}\in\mathbb{N}_0^L\\|\underline{m}|=M,\,|\underline{n}|=N}}\prod_{l=1}^L\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\}(L+2)\,C_\chi^L\,C_t\,\rho_1^{(1+\mu)(M+N)-L}\,\frac{\|w_{m_l+p_l,n_l+q_l}\|_\mu^\#}{\sqrt{p_l^{p_l}q_l^{q_l}}}\\
&\le\sum_{L=1}^\infty(L+2)\left(\frac{C_\chi}{\rho_1}\right)^L C_t\left(2\rho_1^{(1+\mu)}\right)^{M+N}\sum_{\substack{\underline{m},\underline{p},\underline{n},\underline{q}\in\mathbb{N}_0^L\\|\underline{m}|=M,\,|\underline{n}|=N}}\prod_{l=1}^L\left\{\left(\frac{2}{\sqrt{p_l}}\right)^{p_l}\left(\frac{2}{\sqrt{q_l}}\right)^{q_l}\|w_{m_l+p_l,n_l+q_l}\|_\mu^\#\right\}.\end{aligned}$$

Inserting this inequality into the norm $\|\cdot\|^{\#}_{\mu,\xi}$ we obtain the following bound

$$
\begin{aligned}
&\|(w^{(2,\rho_1)}_{M,N})_{M+N\geq 1}\|^{\#}_{\mu,\xi} \\
&\qquad \leq C_t \sum_{M+N\geq 1} \xi^{-(M+N)} \|w^{(2,\rho_1)}_{M,N}\|^{\#}_{\mu} \\
&\qquad \leq 2\, C_t\, \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L+2) \left(\frac{C_\chi}{\rho_1}\right)^L \\
&\qquad\qquad \sum_{M+N\geq 1} \sum_{\substack{\underline{m},\underline{p},\underline{n},\underline{q}\in\mathbb{N}_0^L \\ |\underline{m}|=M,\,|\underline{n}|=N}} \prod_{l=1}^{L} \left\{ \left(\frac{2\xi}{\sqrt{p_l}}\right)^{p_l} \left(\frac{2\xi}{\sqrt{q_l}}\right)^{q_l} \xi^{-(m_l+p_l+n_l+q_l)} \|w_{m_l+p_l,n_l+q_l}\|^{\#}_{\mu} \right\} \\
&\qquad \leq 2\, C_t\, \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L+2) \left(\frac{C_\chi}{\rho_1}\right)^L \left\{ \sum_{m,n,p,q\in\mathbb{N}_0} \left(\frac{2\xi}{\sqrt{p}}\right)^{p} \left(\frac{2\xi}{\sqrt{q}}\right)^{q} \xi^{-(m+p+n+q)} \|w_{m+p,n+q}\|^{\#}_{\mu} \right\}^L \\
&\qquad \leq 2\, C_t\, \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L+2) \left(\frac{C_\chi}{\rho_1}\right)^L \left\{ \sum_{m,n\in\mathbb{N}_0} \left(\sum_{p=0}^{m} \left(\frac{2\xi}{\sqrt{p}}\right)^{p}\right) \left(\sum_{q=0}^{m} \left(\frac{2\xi}{\sqrt{q}}\right)^{q}\right) \xi^{-(m+n)} \|w_{m,n}\|^{\#}_{\mu} \right\}^L \\
&\qquad \leq 2\, C_t\, \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L+2) \left(\frac{C_\chi}{\rho_1}\right)^L 4^L \left(\|w\|^{\#}_{\mu,\xi}\right)^L , \qquad (4.73)
\end{aligned}
$$

where in the second last inequality we used a substitution of summation variables and in the last inequality we used that $\sum_{p=0}^{\infty}\left(\frac{2\xi}{\sqrt{p}}\right)^p \leq \sum_{p=0}^{\infty}(2\xi)^p = \frac{1}{1-2\xi} \leq 2$ since $0<\xi\leq 1/4$. By the assumptions of the theorem we have

$$\|w\|^{\#}_{\mu,\xi} \leq \gamma\,.$$

Inserting this into Eq. (4.73) we find

$$
\begin{aligned}
\big\|(w^{(2,\rho_1)}_{M,N})_{M+N\geq 1}\big\|^{\#}_{\mu,\xi} &\leq 2\, C_t\, \rho_1^{(1+\mu)} \sum_{L=1}^{\infty}(L+2)\left(\frac{4C_\chi\gamma}{\rho_1}\right)^L \\
&\leq 24\, C_t\, \rho_1^{(1+\mu)}\, \frac{C_\chi\gamma}{\rho_1}\Big(1-\frac{4C_\chi\gamma}{\rho_1}\Big)^{-2}, \qquad (4.74)
\end{aligned}
$$

where we used that by assumption

$$0 \leq \frac{4C_\chi\gamma}{\rho_1} < 1\,,$$

and moreover that for $a\in(0,1)$ the following holds

$$\sum_{L=1}^{\infty}(L+2)\,a^L = \sum_{L=3}^{\infty} L\,a^{L-2} = a^{-1}\frac{d}{da}\sum_{L=3}^{\infty} a^L = a^{-1}\frac{d}{da}\frac{a^3}{1-a} \leq \frac{3a}{(1-a)^2}\,.$$

It remains to estimate Eq. (4.65). To this end we recall that for $\underline{m}=\underline{n}=\underline{0}$ we obtain from Lemma 4.1.23

$$\rho_1^{-1}\big\|V^{(1,\rho_1)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[t,w]\big\|^{\#}_{\mu} \leq (L+2)\, C_\chi^L\, C_t\, \rho_1^{-L} \prod_{l=1}^{L} \frac{\|w_{p_l,q_l}\|^{\#}_{\mu}}{\sqrt{p_l^{p_l} q_l^{q_l}}}\,. \qquad (4.75)$$

Using this we find for the derivative

$$
\begin{aligned}
\sup_{r\in[0,1]}\big|\partial_r \hat{w}^{(2,\rho_1)}_{0,0}(r)-1\big| &= \sup_{r\in[0,1]}\Big|\rho_1^{-1}\,\partial_r t(\rho_1 r) + \rho_1^{-1}\Big(\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\underline{p},\underline{q}\in\mathbb{N}_0^L}\partial_r V^{(1,\rho_1)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[t,w](r)\Big) - 1\Big| \\
&\leq \|t'-1\|_\infty + \rho_1^{-1}\sum_{L=1}^{\infty}\sum_{\underline{p},\underline{q}\in\mathbb{N}_0^L}\big\|V^{(1,\rho_1)}_{\underline{0},\underline{p},\underline{0},\underline{q}}[t,w]\big\|^{\#}_{\mu} \\
&\leq \tau_1 + \sum_{L=1}^{\infty}(L+2)\,C_\chi^L\,C_t\,\rho_1^{-L}\sum_{\underline{p},\underline{q}\in\mathbb{N}_0^L}\prod_{l=1}^{L}\frac{\|w_{p_l,q_l}\|^{\#}_{\mu}}{\sqrt{p_l^{p_l}q_l^{q_l}}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \tau_1 + \sum_{L=1}^{\infty} (L+2)\, C_\chi^L \, C_t \, \rho_1^{-L} \left(\sum_{p,q \in \mathbb{N}_0} \frac{\|w_{p,q}\|_\mu^\#}{\sqrt{p^p q^q}} \right)^L \\
&\le \tau_1 + \sum_{L=1}^{\infty} (L+2)\, C_\chi^L \, C_t \, \rho_1^{-L} \left[\|(w_{m,n})_{m+n\ge 0}\|_{\mu,\xi}^\# \right]^L \\
&\le \tau_1 + \sum_{L=1}^{\infty} (L+2)\, C_t \left(\frac{C_\chi \gamma}{\rho_1} \right)^L \\
&\le \rho_1^{-1} \tau_1 + 3\, C_t \, \frac{C_\chi \gamma}{\rho_1} \Big(1 - \frac{C_\chi \gamma}{\rho_1} \Big)^{-2} ,
\end{aligned}
\tag{4.76}
$$

where we used again that by assumption

$$
0 \le \frac{C_\chi \gamma}{\rho_1} < 1 . \tag{4.77}
$$

Analogously we estimate

$$
\begin{aligned}
\big| \hat{w}_{0,0}^{(2,\rho_1)}(0) - \rho_1^{-1} t(0) \big| &\le \big| \rho_1^{-1} \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\underline{p},\underline{q} \in \mathbb{N}_0^L} V_{\underline{0},\underline{p},\underline{0},\underline{q}}^{(1,\rho_1)}[t,w](0) \big| \\
&\le \rho_1^{-1} \sum_{L=1}^{\infty} \sum_{\underline{p},\underline{q} \in \mathbb{N}_0^L} \big\| V_{\underline{0},\underline{p},\underline{0},\underline{q}}^{(1,\rho_1)}[t,w] \big\|_\infty \\
&\le 3\, C_t \, \frac{C_\chi \gamma}{\rho_1} \Big(1 - \frac{C_\chi \gamma}{\rho_1} \Big)^{-2} ,
\end{aligned}
\tag{4.78}
$$

provided (4.77) holds. The claim now follows from Eqns. (4.74), (4.76) and (4.78). □

4.1.6 Proof of main result for the split-up Spin-Boson model

In this subsection we prove Theorem 4.1.2. In the proof we make use of the following result on analyticity. Observe that we denote in the following by B_R the closed ball with radius R.

Theorem 4.1.25 (Analyticity Theorem of Griesemer-Hasler [66])**.**
Let $\mu > 0$, and let $\rho_{\mathrm{GH}} \in (0,1)$ be sufficiently small. Then, for $\xi \in (0,1)$ sufficiently small, there exist positive constants α_0, β_0 and γ_0 such that the following holds.
Let $\mathcal{V}$ be an open subset of $\mathbb{C}$ and let $E_{\mathrm{at}} : \mathcal{V} \to \mathbb{C}$ be an analytic function. Let $\mathcal{U}$ be an open subset of $\mathbb{C} \times \mathbb{C}$ such that for all $s \in \mathcal{V}$ we have

$$
\{s\} \times B_{\rho_{\mathrm{GH}}}(E_{\mathrm{at}}(s)) \subset \mathcal{U} . \tag{4.79}
$$

Suppose $H(\cdot,\cdot)$ is an $\mathcal{L}(\mathcal{F})$-valued analytic function on $\mathcal{U}$, such that for all $(s,\zeta) \in \mathcal{U}$

$$
H(s,\zeta) - (E_{\mathrm{at}}(s) - \zeta) \in \mathcal{B}^{[1]}(\alpha_0, \beta_0, \gamma_0) .
$$

Then there exist analytic functions $\zeta_\infty : \mathcal{V} \to B_{\rho_{\mathrm{GH}}}(s)$ and $\psi_\infty : \mathcal{V} \to \mathcal{H}_{\mathrm{red}}$, nowhere vanishing, such that for all $s \in \mathcal{V}$

$$
H(s, \zeta_\infty(s)) \psi_\infty(s) = 0 .
$$

If furthermore $H(s,\zeta)^* = H(\overline{s}, \overline{\zeta})$ on $\mathcal{U}$, then for real $s \in \mathcal{V}$ the operator $H(s,\lambda)$ has a bounded inverse for all $\lambda \in \big(E_{\mathrm{at}}(s) - \rho_{GH}, \zeta_\infty(s)\big)$.

The proof of Theorem 4.1.25 follows directly from the proof of Theorem 1 in [66, pp. 610-611]. However with the difference that the theorem given above does not involve the initial Feshbach step and has therefore a shorter proof. In the application of Theorem 4.1.25, which we have in mind, the parameter s in the theorem is played by the coupling constant g. Furthermore the energy cutoff ρ_{GH} is an order one quantity, i.e., independent of the coupling constant.

Now we are ready to prove the main result of this section.

Proof of Theorem 4.1.2. Fix $\mu > 0$. Let $\rho_{\mathrm{GH}} \in (0, 1/2]$ and $\xi \in (0, 1/4]$ be chosen sufficiently small such that the assertion of Theorem 4.1.25 holds. The idea is to choose energy cutoffs, ρ_0 and ρ_1, at the first and second Feshbach step such that, for g in a sectorial region of an annulus, we can apply Theorem 4.1.25. As we let the outer and inner radius of the annulus tend to zero we obtain the desired result.

We choose $\epsilon \in (0, \mu)$ and $\alpha \in (0, \min(\mu - \epsilon, 1))$. For $\rho_0 > 0$ we define

$$\rho_1 := \rho_0^{1+2\epsilon+\alpha} . \tag{4.80}$$

Moreover we assume that

$$0 < \rho_0 < \frac{1}{4} . \tag{4.81}$$

We consider the following sectorial region of an annulus, determined by the conditions $g \in S_{\delta_0}$ and

$$c_{\delta_0}^{-1/2}\, \rho_0^{1+\epsilon+\frac{\alpha}{2}} < |g| < \min\left(\rho_0^{1+\epsilon}, \left(8\, \|Z_{\mathrm{at}}\| + 4\, c_{\delta_0}\right)^{-1} \rho_0^{1/2}, \left(8\, \xi^{-1}\, C_F\, \|G\|_\mu\right)^{-1} \rho_0^{1/2}\right) . \tag{4.82}$$

In view of the upper bound in Eq. (4.82), we conclude from Theorem 4.1.9 (Banach space estimate for 1st Feshbach operator) that there exists a finite constant $C^{(1)}$ such that for ρ_0 satisfying (4.81) and all $g \in S_{\delta_0}$ obeying (4.82) and $z \in D_{\rho_0/2}(\epsilon_{\mathrm{at}})$ we have

$$H_g^{(1,\rho_0)}(z) - \rho_0^{-1}(\epsilon_{\mathrm{at}} - z) \in \mathcal{B}^{[d_0]}\big(C^{(1)} \rho_0^{\frac{1}{2}+\epsilon}, C^{(1)} \rho_0^{\frac{1}{2}+\epsilon}, C^{(1)} \rho_0^{\mu+1+\epsilon}\big) . \tag{4.83}$$

Next we want to use Theorem 4.1.19 (Feshbach pair criterion for 2nd iteration). In order to do that we observe that by Eqns. (4.80) and (4.81) the assumptions on the ρ's and the Assumptions (4.53) and (4.54) are satisfied for $g \in S_{\delta_0}$ with (4.82). To apply the theorem we consider the decomposition given in Eq. (4.57), i.e.

$$H_g^{(1,\rho_0)}(z) = T_g^{(1,\rho_0)}(z) + W_g^{(1,\rho_0)}(z) .$$

We apply Lemma 4.1.15 (Free approximation to 1st Feshbach operator) and see that the difference to the free approximation, Eq. (4.60), is strictly smaller than $C\rho_0^2$ for some constant C. More precisely,

$$\|t_g^{(1,\rho_0)}(z) - t_{g,\mathrm{free}}^{(1,\rho_0)}(z)\| \le C\rho_0^{2+2\epsilon} , \tag{4.84}$$

$$\|P_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)}\, \overline{P}_{\mathrm{at}}^{(2)} + \overline{P}_{\mathrm{at}}^{(2)} w_{g,0,0}^{(1,\rho_0)} P_{\mathrm{at}}^{(2)}\| \le C\rho_0^{2+2\epsilon} , \tag{4.85}$$

where we used the upper bound in Eq. (4.82). From the definition in Eq. (4.80) we see that the right hand side of Eq. (4.84) is less than $\rho_1/4$ provided ρ_0 is sufficiently small. Thus by Neumann's theorem (Theorem 3.2.9) and Lemma 4.1.17 we see that $T_g^{(1,\rho)}(z)$ is invertible on $\mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}$ and

$$\|(H_{0,0}(t_g^{(1,\rho_0)}) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)})^{-1}\| \le \frac{8}{\rho_1} .$$

Furthermore, we get from Eq. (4.83) and (4.85) that

$$\|w_{g,\mathrm{int}}^{(1,\rho_0)}(z)\|_{\mu,\xi}^{\#} \le C\rho_0^{2+2\epsilon} + C^{(1)} \rho_0^{\mu+1+\epsilon} =: \gamma .$$

We note that

$$\frac{\gamma}{\rho_1} = C\rho_0^{1-\alpha} + C^{(1)} \rho_0^{\mu-\epsilon-\alpha} ,$$

tends to zero for $\rho_0 \to 0$. In view of Eq. (3.13) we see that Condition (4.61) of Theorem 4.1.19 is satisfied for ρ_0 small. Thus $T_g^{(1,\rho_0)}(z)$ and $W_g^{(1,\rho_0)}(z)$ are a Feshbach pair for $\boldsymbol{\chi}_{\rho_1}^{(1)}$, provided $g \in S_{\delta_0}$ obeys (4.82) and $z \in D_{\rho_0\rho_1/2}(\epsilon_{\mathrm{at}} + g^2 \epsilon_{\mathrm{at}}^{(2)})$. It now follows from Theorem 4.1.21 (Banach space estimate for 2nd Feshbach operator) that for some constant $C^{(2)}$ we have

$$\begin{aligned} S_{\rho_1}\Big(F_{\boldsymbol{\chi}_{\rho_1}^{(1)}}\big(T_g^{(1,\rho_0)}(z), W_g^{(1,\rho_0)}(z)\big)\Big) - \rho_1^{-1}\rho_0^{-1}(\epsilon_{\mathrm{at}} - z + g^2 \epsilon_{\mathrm{at}}^{(2)}) \\ \in \mathcal{B}^{[1]}\big(C^{(2)} \frac{\gamma}{\rho_1}, C^{(2)} \rho_0^{\frac{1}{2}+\epsilon} + C^{(2)} \frac{\gamma}{\rho_1}, C^{(2)} \rho_1^{\mu} \gamma\big) . \end{aligned} \tag{4.86}$$

for all $z \in D_{\rho_0\rho_1/2}(\epsilon_{\mathrm{at}} + g^2 \epsilon_{\mathrm{at}}^{(2)})$ and $g \in S_{\delta_0}$ with (4.82).

Now we aim to apply Theorem 4.1.25 (Analyticity Theorem of Griesemer-Hasler). To this end we express the Hamiltonian in terms of the variables $\zeta := \rho_0^{-1}\rho_1^{-1}z$ and $s := g$. Moreover we define the function $E_{\rm at}(s) := \rho_1^{-1}\rho_0^{-1}(\epsilon_{\rm at} + s^2\epsilon_{\rm at}^{(2)})$ on $\mathbb{C}$ and the function

$$H(s,\zeta) := S_{\rho_1}\left(F_{\boldsymbol{\chi}_{\rho_1}^{(1)}}\left(T_s^{(1,\rho_0)}(\rho_0\rho_1\zeta), W_s^{(1,\rho_0)}(\rho_0\rho_1\zeta)\right)\right),$$

for $(s,\zeta) \in \mathcal{U} := \bigcup_{g\in S_{\delta_0}:(4.82)} D_{1/2}(E_{\rm at}(g))$. Since we have expressed $H(s,\zeta)$ in terms of uniformly convergent Neumann series (Theorems 4.1.6 and Theorem 4.1.19), we can conclude that $H(s,\zeta)$ is jointly analytic on $\mathcal{U}$. In addition, the conjugation property $H(s,\zeta)^* = H(\overline{s},\overline{\zeta})$ holds because each term in the convergent expansion has that property. Moreover, Condition (4.79) holds if ρ_{GH} is less than 1/2. Hence we can conclude from Theorem 4.1.25 that there exist analytic functions

$$g \mapsto \zeta_\infty(g), \qquad g \mapsto \psi_\infty(g),$$

for $g \in S_{\delta_0}$ with (4.82) such that

$$H(g,\zeta_\infty(g))\psi_\infty(g) = 0,$$

and additionally $\zeta_\infty(g) \in D_{1/2}(E_{\rm at}(g))$. Expressed in terms of the original variables we obtain from the isospectrality of the Feshbach map, that

$$E_g := \rho_0\rho_1\zeta_\infty(g) \quad \text{and} \quad \psi_g := Q_g^{(0,\rho_0)}(E_g)Q_g^{(1,\rho_1)}(E_g)\psi_\infty(g),$$

are an eigenvalue and eigenvector of H_g, where we use the abbreviations

$$Q_g^{(0,\rho_0)}(z) := \boldsymbol{\chi}_{\rho_0}^{(0)} - \overline{\boldsymbol{\chi}}_{\rho_0}^{(0)}\left(H_0 - z + \overline{\boldsymbol{\chi}}_{\rho_0}^{(0)} gW \overline{\boldsymbol{\chi}}_{\rho_0}^{(0)}\right)^{-1} gW\boldsymbol{\chi}_{\rho_0}^{(0)},$$
$$Q_g^{(1,\rho_1)}(z) := \boldsymbol{\chi}_{\rho_1}^{(1)} - \overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}\left(T_g^{(1,\rho_0)}(z) + \overline{\boldsymbol{\chi}}_{\rho_1}^{(1)} W_g^{(1,\rho_0)}(z)\overline{\boldsymbol{\chi}}_{\rho_1}^{(1)}\right)^{-1} W_g^{(1,\rho_0)}\boldsymbol{\chi}_{\rho_1}^{(1)}.$$

It now follows that E_g and ψ_g are analytic functions of $g \in S_{\delta_0}$ with (4.82) since $Q_g^{(0,\rho_0)}(z)$ and $Q_g^{(1,\rho_1)}(z)$ are analytic functions of g and z, as they are given by convergent expansions of jointly analytic functions. Furthermore in terms of the original spectral parameter we have $E_g \in D_{\rho_0\rho_1/2}(\epsilon_{\rm at} + g^2\epsilon_{\rm at}^{(2)})$, which implies Eq. (4.6). The conjugation property of $H(s,\zeta)$, the last statement in Theorem 4.1.25, and the isospectrality property of the Feshbach map imply that E_g is the ground-state energy of H_g. Therefore as we take ρ_0 to zero, the theorem follows. □

We note that the choice for ρ_0 and ρ_1 in the proof corresponds to ρ_0 being larger than $|g|$ and ρ_1 being smaller than $|g|$ such that their product is smaller than $|g|^2$.

Remark 4.1.26. In this section we considered a quantum mechanical system that possesses a degeneracy of the ground-state eigenvalue. We assumed that this degeneracy is lifted at second order in formal perturbation theory. Under these conditions we showed that the ground state as well as the ground-state energy are analytic as functions of the coupling constant in an open cone with apex at the origin. We believe that the methods used in this section are also useful to treat degeneracies which are lifted at an order higher than second order, by possibly inserting several Feshbach projections in between, with energy cutoffs depending on the coupling constant.

To conclude this section we remark on *Borel summability*.

Remark 4.1.27. Borel summability methods allow in certain situations to recover a function from its asymptotic expansion (cf. Chapter 5), provided it satisfies a strong asymptotic condition. Theorem 4.1.2 together with Remark 4.1.4 can be used to show that the ground-state energy as a function of g^2 satisfies the analyticity requirement of a strong asymptotic condition [115]. Suppose $\pi/4 < \delta_0 < \pi/2$ and g_0 are as in Theorem 4.1.2. Define the function $f(w) := E_{\sqrt{w}}$ with $|\arg(w)| < 2\delta_0$ and $|\sqrt{w}| < g_0$. Then f is analytic in the interior of the cone $S_{2\delta_0} \cap D_{g_0^2}$ and extends continuously onto the boundary. Moreover f satisfies the analyticity requirement for a strong asymptotic condition. Now suppose there are C and σ such that

$$\left| f(w) - \sum_{n=0}^{N} c_n w^n \right| \leq C\sigma^{N+1}(N+1)!\,|w|^{N+1}, \tag{4.87}$$

for all N and all $w \in S_{2\delta_0} \cap D_{g_0^2}$. Then $E_g = f(g^2)$ can be recovered uniquely by the method of Borel summability [115, Watson's theorem].

4.2 Symmetries for the Spin-Boson Hamiltonian

In this section we consider a quantum mechanical system that is invariant under certain unitary and antiunitary linear transformations. Such symmetries (Definition 4.2.10) may cause degeneracies in the spectrum of operators corresponding to observables of the system (Lemma 4.2.12). Examples are the time reversal and parity symmetry. Time reversal symmetry corresponds to an anti-linear transformation implementing $t \to -t$ and parity symmetry to a linear transformation implementing the reversal of sign on all spatial coordinates. We concentrate our analysis on the ground state and the ground-state eigenvalue of a perturbed Spin-Boson Hamiltonian. More precisely, we consider a system of matter particles that is described by a closed symmetric operator acting on a separable Hilbert space. We assume that this 'atomic system' interacts with a quantized field of massless Bosons by means of a linear coupling. The field of massless Bosons is modeled by the bosonic Fock space. Hence the Hamiltonian describing the total system is a generalized Spin-Boson Hamiltonian (cf. Chapter 2).
As in Section 4.1 the ground-state eigenvalue of the atomic Hamiltonian is assumed to be degenerate. In this section however, the degeneracy is caused by a set of symmetries that can be represented in a certain way (cf. Hypothesis II (ii) in Subsection 4.2.2). This guarantees that, if the ground-state eigenvalue is degenerate than this degeneracy persists once the interaction is turned on, i.e. *the degeneracy is protected by the set of symmetries.* Moreover we assume that the interaction satisfies an infrared condition (Hypothesis I). The infrared condition is needed for the renormalization analysis to converge. We show that the ground state exists for small values of the coupling constant, which is a well-known result [60, 67, 99, 127]. Furthermore, we show that the ground state as well as the ground-state eigenvalue are analytic as functions of the coupling constant. This result complements the result in [66] where only the non-degenerate situation was considered.

4.2.1 Preliminaries

In this subsection we recall basic definitions and state results which we later use. In particular we define what we mean by a symmetry of an operator and prove two versions of Schurs lemma. These lemmas are crucial for the proof of the main theorem (Theorem 4.2.26). Moreover we consider parameter-dependent linear operators and present general analyticity results for analytic families of operators.

Symmetry operators, Schurs Lemma and Kramers' theorem

Definition 4.2.1. A bounded linear operator T is said to be *isometric* if it preserves the norm, i.e.

$$\|Tu\|_{\mathcal{H}_2} = \|u\|_{\mathcal{H}_1}$$

for every $u \in \mathcal{H}_1$.

Remark 4.2.2. An isometric operator T is injective and has the following properties:

(i) $T^*T = \mathbb{1}_{\mathcal{H}_1}$,

(ii) $\langle Tx, Ty\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1}$ for all $x,y \in \mathcal{H}_1$.

Definition 4.2.3. An isometric operator T is said to be *unitary* if it is surjective that is, if T has range $\mathcal{H}_2$.

Remark 4.2.4. If T is unitary, then $T^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ is a bounded linear operator and is itself unitary. We therefore get the following equivalences

$$T : \mathcal{H}_1 \to \mathcal{H}_2 \text{ is unitary} \iff T^*T = \mathbb{1}_{\mathcal{H}_1} \text{ and } TT^* = \mathbb{1}_{\mathcal{H}_2} \iff T^{-1} = T^*.$$

Definition 4.2.5. An antilinear operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ that is surjective and satisfies

$$\langle Ux, Uy\rangle_{\mathcal{H}_2} = \overline{\langle x, y\rangle}_{\mathcal{H}_1} ,$$

for all $x, y \in \mathcal{H}_1$ is called an *antiunitary operator.*

Remark 4.2.6. The definition of the adjoint (Definition 2.1.8) for an antiunitary operator U has to be changed into

$$\langle Ux, y\rangle = \overline{\langle x, U^*y\rangle} , \quad \text{for } x, y \in \mathcal{H}_1 .$$

Moreover the adjoint of an antiunitary U is also antiunitary and for $\mathcal{H}_1 = \mathcal{H}_2$ we have $UU^* = U^*U = \mathbb{1}_{\mathcal{H}_1}$.

Recall that we omit subscripts on norms and inner products whenever it is clear from the context on which spaces the norms and inner products act.

Definition 4.2.7. Let V be a subspace of a Hilbert space $\mathcal{H}$ and let $\mathcal{S}$ be a set whose elements are unitary or antiunitary operators on $\mathcal{H}$. We say that $S \in \mathcal{S}$ acts *irreducibly* on V if for any subspace W of V with $SW \subset W$ we have $W = \{0\}$ or $W = V$.

The next two lemmas are versions of the well-known Lemma of Schur [121]. The first one is for self-adjoint operators. Since the perturbed Hamiltonian is in general non-self-adjoint we present a second one for ordinary linear operators, as well.

Lemma 4.2.8. *Let $\mathcal{S}$ be a set containing unitary and antiunitary operators which act irreducibly on a complex finite-dimensional Hilbert space V. Let T be a self-adjoint linear operator on V such that*

$$S^*TS = T\,, \qquad \text{for all } S \in \mathcal{S}.$$

Then there exists a number $\lambda \in \mathbb{R}$ such that $T = \lambda\, \mathbb{1}_V$.

Proof. First observe that T has a real eigenvalue, say λ. Thus $T - \lambda$ has a nonvanishing kernel. Now S leaves the space $\mathrm{Ker}(T - \lambda)$ invariant since λ is real. Thus by irreducibility we see that $\mathrm{Ker}(T - \lambda) = V$. This yields the claim. □

Now we want to extend the above lemma to non-self-adjoint operators.

Lemma 4.2.9. *Let $\mathcal{S}$ be a set containing unitary and antiunitary operators which act irreducibly on a complex finite-dimensional Hilbert space V. Let T be a linear operator on V such that*

$$\begin{aligned} S^*TS &= T\,, \quad \text{for all } S \in \mathcal{S},\ S \text{ unitary}, \\ S^*TS &= T^*\,, \quad \text{for all } S \in \mathcal{S},\ S \text{ antiunitary}. \end{aligned} \tag{4.88}$$

Then there exists a number $\lambda \in \mathbb{C}$ such that $T = \lambda\, \mathbb{1}_V$.

Proof. Note that there exits a unique decomposition

$$T = Z + iY\,,$$

with Y and Z self-adjoint operators on V. Then it follows from Eq. (4.88) that

$$S^*ZS = Z\,, \quad S^*YS = Y\,,$$

for all $S \in \mathcal{S}$. Thus Z and Y are multiples of the identity by Lemma 4.2.8. □

For notational simplicity we define

Definition 4.2.10. A unitary or antiunitary operator S is a *symmetry of the operator T*, if

$$\begin{aligned} S^*TS &= T\,, \quad \text{for } S \text{ unitary}, \\ S^*TS &= T^*\,, \quad \text{for } S \text{ antiunitary}. \end{aligned}$$

Remark 4.2.11. The set of symmetries of an operator in a complex Hilbert space form a group.

The subsequent lemma illustrates how a symmetry of a self-adjoint operator can induce degeneracy for the eigenvalues of this operator. It is an abstract version of Kramers' degeneracy. This version is from [106] and we note that the same proof can also be found in [104].

Lemma 4.2.12. *(Kramers' degeneracy theorem) Let an antiunitary operator θ be a symmetry of a self-adjoint operator H and assume $\theta^2 = -1$. Then each eigenvalue of H is at least two-fold degenerate.*

Proof. Let $\psi \neq 0$ be an eigenvector of H with eigenvalue E. We claim that $\theta\psi$ is an eigenvector of H which is orthogonal to ψ. Clearly $\theta\psi \neq 0$

$$H\theta\psi = \theta H\psi = \theta E\psi = E\theta\psi\,.$$

On the other hand

$$\langle \psi, \theta\psi \rangle = \langle \theta\theta\psi, \theta\psi \rangle = -\langle \psi, \theta\psi \rangle\,.$$

Thus the inner product has to vanish. □

Parameter dependence and analyticity

Let $\mathcal{H}$ be a separable Hilbert space and $\nu \in \mathbb{N}$. In the previous chapters we already saw examples of linear operator that depend on real or complex parameters (cf. Eq. (2.13) and (3.2)). In the following we gather parameter-dependent notations and state results for analytic families of operators.

Definition 4.2.13. A (possibly unbounded) operator-valued function $T(s)$ on a complex domain X is called an *analytic family* if and only if:

(i) for each $s \in X$, $T(s)$ is closed and has a nonempty resolvent set,

(ii) for every $s_0 \in X$, there is a $\lambda_0 \in \rho(T(s_0))$ such that $\lambda_0 \in \rho(T(s))$ for s near s_0 and $(T(s) - \lambda_0)^{-1}$ is an analytic operator-valued function of s near s_0.

Remark 4.2.14. An analytic family is also called *analytic family in the sense of Kato.*

Definition 4.2.15. Let R be a connected domain in the complex plane and let $T(s)$, a closed operator with nonempty resolvent set, be given for each $s \in R$. We say that the operator $T(s)$ is an *analytic family of type (A)* if and only if

(i) the operator domain of $T(s)$ is some set D independent of s,

(ii) for every $\psi \in D$, $T\psi$ is a vector-valued analytic function of s.

Remark 4.2.16. Every analytic family of type (A) is an analytic family in the sense of Kato, cf. [115]. For a detailed description of the theory of analytic functions we refer the reader to [81, 93].

Definition 4.2.17. Let $X \subset \mathbb{C}^\nu$ and $s \in X$.

(a) Let $\mathcal{S}$ be a set of unitary and antiunitary operators. The map $X \to \mathcal{L}(\mathcal{H})$; $s \mapsto T(s)$ is said to *commute* with the set $\mathcal{S}$ if every $S \in \mathcal{S}$ is a symmetry of the operator $T(s)$ for all $s \in X$.

(b) Let $\overline{X} = X$, we say that the map $X \to \mathcal{L}(\mathcal{H})$; $s \mapsto T(s)$ is *reflection symmetric* if

$$T(s)^* = T(\overline{s}), \qquad \text{for all } s \in X.$$

Remark 4.2.18. The following two lemmas play a significant role later in this section.

Lemma 4.2.19. *Let $\overline{X} = X \subset \mathbb{C}^\nu$. Let $\mathcal{S}$ be a set of unitary and antiunitary operators acting irreducibly on a finite-dimensional Hilbert space V. Suppose the map $X \to \mathcal{L}(V)$; $s \mapsto T(s)$ is reflection symmetric and commutes with the set $\mathcal{S}$. Then there exists a function $f : X \to \mathbb{C}^\nu$ such that $\overline{f(s)} = f(\overline{s})$ and*

$$T(s) = \mathbb{1}_V f(s).$$

Proof. For fixed $s \in X$ every $S \in \mathcal{S}$ is a symmetry for $T(s)$. Hence by Lemma 4.2.9 there exists a function $f : X \to \mathbb{C}^\nu$ such that $T(s) = \mathbb{1}_V f(s)$. Moreover, since $T(s)$ is reflection symmetric, this function satisfies $\overline{f(s)} = f(\overline{s})$. □

Lemma 4.2.20. *Let $\mathcal{H}$ be a complex Hilbert space and $P(s) \in \mathcal{L}(\mathcal{H})$ a projection-valued analytic function on a connected, simple connected region of the complex plane X containing s_0. Then there exists an analytic family $U(s)$ of bounded and invertible operators with the following properties:*

(a) $U(s)\, P(s_0)\, U(s)^{-1} = P(s)$,

(b) if $s_0 \in \mathbb{R}$ and $P(s)$ is self-adjoint for real $s \in X$, then we can choose $U(s)$ unitary for real s.

(c) if $P(s)$ commutes with a finite set of symmetries $\mathcal{S}$, then $U(s)$ satisfies

(i) $S^* U(s) S = U(s)$ *for unitary* $S \in \mathcal{S}$,
(ii) $S^* U(s) S = \left(U(s)^{-1}\right)^*$ *for antiunitary* $S \in \mathcal{S}$.

Moreover, if in addition $\overline{X} = X$, $X \cap \mathbb{R} \neq \emptyset$ and $P(s)$ is reflection symmetric for all $s \in X$, then

$$U(\overline{s}) = U(s)^{-1}.$$

Proof. For a proof of (a) and (b) we refer the reader to [115, Theorem XII.12]. In the corresponding proof and a related lemma it is shown that the analytic family $U(s)$ is the unique solution of the differential equation

$$\frac{d}{ds}Z(s) = Q(s)Z(s)\,, \qquad Z(s_0) = \mathbb{1}\,, \tag{4.89}$$

where $s_0 \in X$ and $Q(s) := P'(s)P(s) - P(s)P'(s)$ is the commutator of $P' = \frac{d}{ds}P$ and P. Moreover it is shown that $U(s)^{-1}$ is the unique solution of the differential equation

$$\frac{d}{ds}V(s) = V(s)\big(-Q(s)\big)\,, \qquad V(s_0) = \mathbb{1}\,. \tag{4.90}$$

Now we proof (c):
For unitary $S \in \mathcal{S}$ the statement (i) follows directly from the uniqueness of solutions for Eq. (4.89). More precisely, since $S^*P(s)S = P(s)$ we obtain $\frac{d}{ds}P(s) = S^*\frac{d}{ds}P(s)S$ using the product rule. Hence $S^*Q(s)S = Q(s)$ and the statement follows from the uniqueness of the solution for Eq. (4.89), since

$$\frac{d}{ds}S^*U(s)S = S^*\frac{d}{ds}U(s)S = S^*Q(s)U(s)S = Q(s)(S^*U(s)S)\,,$$

and

$$S^*U(s_0)S = U(s_0) = \mathbb{1}\,.$$

Next we consider antiunitary $S \in \mathcal{S}$. Every such S is a symmetry for $P(s)$ by assumption. Thus, by taking adjoints, we have $S^*P(s)^*S = P(s)$ and differentiating this we obtain $S^*\big(\frac{d}{ds}P(s)\big)^*S = \frac{d}{ds}P(s)$. An easy calculation shows

$$S^*Q(s)^*S = -Q(s)\,. \tag{4.91}$$

Now using the fact that $U(s)$ solves Eq. (4.89) we see that $\big(S^*U(s_0)S\big)^* = \mathbb{1}$ and compute

$$\frac{d}{ds}\big(S^*U(s)S\big)^* = \left(S^*\frac{d}{ds}U(s)S\right)^* = \left(S^*Q(s)U(s)S\right)^* = \big(S^*U^*(s)S\big)\big(-Q(s)\big)\,,$$

where the last inequality follows due to Eq. (4.91). Hence by the uniqueness of solution for Eq. (4.90) we obtain that $S^*U(s)^*S = U(s)^{-1}$, and therefore

$$S^*U(s)S = \big(U(s)^{-1}\big)^*\,.$$

In order to proof the last part we first consider $s \in \mathbb{R} \cap X$. Since $P(s)$ is by assumption self-adjoint for real $s \in X$ we can choose by part (b) the analytic family $U(s)$ unitary for real s. Hence $U(\overline{s})^* = U(s)^{-1}$. Now, using analytic continuation, we extend this to all $s \in X$. $\square$

Last but not least we consider a uniformly bounded family of operator-valued functions that depend on some parameter s. The following lemma shows for which conditions this family is analytic in s. For a proof of this lemma we refer to [66, Appendix B].

Lemma 4.2.21. *Suppose the function $F : X \to \mathcal{L}(\mathcal{H}_{\mathrm{at}}; L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{H}_{\mathrm{at}}))$, $s \mapsto F_s$ is uniformly bounded and suppose for almost every $k \in \mathbb{R}^3 \times \mathbb{Z}_2$ and all $s \in X$, there exists an operator $F_s(k) \in \mathcal{L}(\mathcal{H}_{\mathrm{at}})$ such that $F_s(k)\varphi = (F_s\varphi)(k)$ for all $\varphi \in \mathcal{H}_{\mathrm{at}}$. If for almost every $k \in \mathbb{R}^3 \times \mathbb{Z}_2$, the function $s \mapsto F_s(k) \in \mathcal{L}(\mathcal{H}_{\mathrm{at}})$ is analytic, then F is analytic.*

4.2.2 Description of the model and statement of result

We begin with a detailed definition of the considered model. Especially we formulate three Hypothesis that contain distinctive properties of our model. In the end we state the main result of this section.
We already mentioned in Chapter 2 that the considered model is a specific case of Example 2.3.5. In particular we consider a model that is similar to the model in [66].

Let $X \subseteq \mathbb{C}^\nu$, $\nu \in \mathbb{N}$, be an open set that is symmetric with respect to complex conjugation and satisfies $X \cap \mathbb{R}^\nu \neq \emptyset$, i.e. $\overline{X} = X$. Moreover let $\mathcal{H}_{\mathrm{at}}$ be an arbitrary, separable complex Hilbert space and $\mathcal{F}$ the bosonic Fock space defined in Chapter 2. On $\mathcal{H} := \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}$ we define a family of unbounded operators

$$H_g(s) : D(H_g(s)) \subset \mathcal{H} \to \mathcal{H}\,,$$

for $s \in X$ and the coupling constant $g \geq 0$ by

$$H_g(s) := H_{\mathrm{at}}(s) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}_{\mathrm{at}}} \otimes H_f + gW(s)\,.$$

We assume that the atomic Hamiltonian $H_{\mathrm{at}}(s)$ is closed and symmetric on $\mathcal{H}_{\mathrm{at}}$. As usual we denote by H_f the free field Hamiltonian with dispersion relation $\omega(k) := |k|$. For more details see Eq. (2.7).
The interaction operator

$$W(s) := a(\omega^{-1/2} G_{\overline{s}}) + a^*(\omega^{-1/2} G_s) \tag{4.92}$$

is defined as the sum of a smeared annihilation and creation operator with $G_s \in \mathcal{L}(\mathcal{H}_{\mathrm{at}}; \mathcal{H}_{\mathrm{at}} \otimes \mathfrak{h})$.

Remark 4.2.22. In Section 3.3 we defined smeared annihilation and creation operators for such operator-valued coupling functions G. Especially we refer to Eq. (3.16) and Remark 3.3.9 for more information.

In the following we formulate three Hypotheses. These Hypotheses are crucial for our proof of the main theorem of this section (Theorem 4.2.26). Each Hypothesis addresses a different property of our model. For example, we need some infrared regularity for the renormalization analysis to be applicable. Hence we assume a specific infrared condition in Hypothesis I. As in Section 4.1 we define this infrared condition in terms of the norm on the space

$$L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}})) := \{G : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{L}(\mathcal{H}_{\mathrm{at}}) : G \text{ measurable}, \|G\|_\mu < \infty\}\,,$$

where $\mu > 0$ and the norm $\|\cdot\|_\mu$ is given by Eq. (4.5). Now we can formulate the first Hypothesis.

Hypothesis I. For $s \in X$ the mapping $s \mapsto G_s$ is a bounded analytic function that has values in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$. Moreover there exists a $\mu > 0$ such that

$$\sup_{s \in X} \|G_s\|_\mu < \infty\,.$$

A direct consequence of this Hypothesis is that the interaction operators $W(s)$ and its adjoint $W(s)^*$ are well-defined operators on $\mathcal{H}_{\mathrm{at}} \otimes D(H_f)$ and infinitesimally bounded with respect to H_f for all $s \in X$. This follows by combining Hypothesis I with Remark 3.3.9 and Lemma 3.3.12 from Chapter 3. Hence the operator $H_g(s)$ is defined on $D\big(H_{\mathrm{at}}(s)\big) \otimes D(H_f)$. Since $H_{\mathrm{at}}(s)$ is closed, this space is dense in $\mathcal{H}$ and $H_g(s)$ is densely defined. Thus the adjoint $H_g(s)^*$ exists and is closed. Moreover, $D\big(H_{\mathrm{at}}(s)\big) \otimes D(H_f)$ is contained in the domain of $H_g(s)^*$. Hence the map $H_g(s) : D\big(H_{\mathrm{at}}(s)\big) \otimes D(H_f) \subset \mathcal{H} \to \mathcal{H}$ has a densely defined adjoint and is therefore closable [93, Theorem 5.28].
Next we state the second Hypothesis and explain its importance in connection with degenerate eigenvalues in the subsequent remark.

Hypothesis II. The mapping $s \mapsto H_{\mathrm{at}}(s)$ is an analytic family of type (A) and $H_{\mathrm{at}}(s)^* = H_{\mathrm{at}}(\overline{s})$ for all $s \in X$. In particular $H_{\mathrm{at}}(s)$ is self-adjoint for $s \in \mathbb{R}^\nu \cap X$. Moreover,

(i) there exists $s_0 \in \mathbb{R}^\nu \cap X$ such that $E_{\mathrm{at}}(s_0) := \inf \sigma(H_{\mathrm{at}}(s_0))$ is a discrete eigenvalue of $H_{\mathrm{at}}(s_0)$.

(ii) the eigenvalue $E_{\mathrm{at}}(s_0)$ is either non-degenerate or if it is degenerate then there exists a set of symmetries, $\mathcal{S}$, commuting with $H_g(s)$ for all g in a real neighborhood of zero and $s \in X$. Each element of $\mathcal{S}$ can be written in the form $S_1 \otimes S_2$, where S_1 is a symmetry in $\mathcal{H}_{\mathrm{at}}$ and S_2 is a symmetry in $\mathcal{F}$ leaving the Fock vacuum invariant. In particular, the set of symmetries in $\mathcal{H}_{\mathrm{at}}$

$$\{S_1 : S_1 \otimes S_2 \in \mathcal{S}\}$$

acts irreducibly on the eigenspace corresponding to the eigenvalue $E_{\mathrm{at}}(s_0)$.

Remark 4.2.23. We note that if Hypothesis II (ii) holds, then W commutes with $\mathcal{S}$, H_{at} commutes with $\{S_1 : S_1 \otimes S_2 \in \mathcal{S}\}$, and H_f commutes with $\{S_2 : S_1 \otimes S_2 \in \mathcal{S}\}$.

The subsequent lemma illustrates how Hypothesis II and in particular the existence of a set of symmetries as in part (ii) simplifies the spectral analysis in the degenerate case. For this let us first examine the implications of part (i). For notational simplicity we set $\nu = 1$. Similar to the first part of the proof of Theorem XII.8 in [115] we obtain, since $E_{\mathrm{at}}(s_0)$ is isolated, that we can pick $\epsilon > 0$ such that

$$\sigma(H_{\mathrm{at}}(s_0)) \cap \big\{z \in \mathbb{C} : |z - E_{\mathrm{at}}(s_0)| < \epsilon\big\} = \big\{E_{\mathrm{at}}(s_0)\big\}\,.$$

In addition, since $s \mapsto H_{\rm at}(s)$ is an analytic family of type (A) on X, the set

$$\Gamma = \big\{(s,\lambda) : s \in X,\, \lambda \in \rho(H_{\rm at}(s))\big\}$$

is open and the function $(H_{\rm at}(s)-z)^{-1}$ defined on Γ is an analytic function of two variables, see Theorem XII.7 in [115]. Hence, since the set $\big\{z \in \mathbb{C} : |z - E_{\rm at}(s_0)| = \epsilon\big\}$ is a compact set in $\mathbb{C}$, we can find for all $s \in X$ an appropriate $\delta > 0$ such that

$$z \notin \sigma(H_{\rm at}(s))\,, \quad \text{if } |z-E_{\rm at}(s_0)| = \epsilon \text{ and } |s-s_0| \le \delta\,.$$

Moreover we obtain that

$$P_{\rm at}(s) = -\frac{1}{2\pi i}\oint_{|z-E_{\rm at}(s_0)|=\epsilon} \frac{1}{H_{\rm at}(s)-z}\,dz \tag{4.93}$$

exists and is analytic for $s \in N := \{s \in X : |s-s_0| \le \delta\}$.
We note that if $H_{\rm at}$ commutes with $\{S_1 : S_1 \otimes S_2 \in \mathcal{S}\}$ then also $P_{\rm at}$ commutes with this set.

Lemma 4.2.24. *Suppose the situation is as in Hypothesis II and let the eigenvalue $E_{\rm at}(s_0)$ be degenerate. Then there exists a neighborhood, N, of s_0 in X and an analytic function $E_{\rm at} : N \to \mathbb{C}$ such that for all $s \in N$*

$$H_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s) = E_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s)\,.$$

Moreover, for fixed $s \in N$, the point $E_{\rm at}(s) \in \mathbb{C}$ is the only eigenvalue of $H_{\rm at}(s)$ in a neighborhood of $E_{\rm at}(s_0)$.

Proof. Using the implications from Hypothesis (i), stated above, we deduce from Theorem XII.6 in [115] that, if $P_{\rm at}(s)$ has dimension n, then $H_{\rm at}(s)$ has at most n points of its spectrum inside the set $\big\{z \in \mathbb{C} : |z-E_{\rm at}(s_0)| < \epsilon\big\}$ and each of these points is discrete. Moreover we have that

$$\sigma\big(H_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s)\big) = \sigma(H_{\rm at}(s)) \cap \big\{z \in \mathbb{C} : |z-E_{\rm at}(s_0)| < \epsilon\big\}\,.$$

In addition we know from Hypothesis II (ii) that the eigenvalue $E_{\rm at}(s_0)$ has at most finite multiplicity, let say n. Hence $\dim\big({\rm Ran}P_{\rm at}(s_0)\big) = n$. In addition using Lemma 4.2.20 we get that there exists an analytic family $U(s)$ of bounded and invertible operators, that are unitary for $s \in \mathbb{R} \cap N$, where $N = \{s \in X : |s-s_0| \le \delta\}$ was defined after Eq. (4.93). Moreover we have

$$U(s)P_{\rm at}(s_0)U(s)^{-1} = P_{\rm at}(s)\,.$$

Hence ${\rm Ran}P_{\rm at}(s_0)$ is an invariant subspace of $U(s)^{-1}H_{\rm at}(s)U(s)$ for $s \in N$ and therefore the operator family

$$H^U_{\rm at}(s) := U(s)^{-1}H_{\rm at}(s)U(s) \restriction {\rm Ran}P_{\rm at}(s_0)$$

is n-dimensional and self-adjoint for $s \in \mathbb{R} \cap N$. Using part (c) of Lemma 4.2.20 we obtain that $H^U_{\rm at}(s)$ commutes with the set $\tilde{\mathcal{S}} = \{S_1 : S_1 \otimes S_2 \in \mathcal{S}\}$. More precisely we have for unitary $S \in \tilde{\mathcal{S}}$ that $S^*H^U_{\rm at}(s)S = S^*U(s)^{-1}SS^*H_{\rm at}(s)SS^*U(s)S = H^U_{\rm at}(s)$. For antiunitary $S \in \tilde{\mathcal{S}}$ we obtain

$$S^*H^U_{\rm at}(s)S = S^*U(s)^{-1}SS^*H_{\rm at}(s)SS^*U(s)S = U(s)^*H_{\rm at}(s)^*\big(U(s)^{-1}\big)^* = H^U_{\rm at}(s)^*\,,$$

where we used for the second equality that we have

$$S^*U(s)^{-1}S = \big(S^*U(s)S\big)^{-1} = \big(\big(U(s)^{-1}\big)^*\big)^{-1} = U(s)^*\,,$$

by part (c) of Lemma 4.2.20. Hence due to Lemma 4.2.19 there exists an analytic function $E_{\rm at} : N \to \mathbb{C}$ such that $H^U_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s_0) = E_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s_0)$. Multiplying with $U(s)$ from the left and $U(s)^{-1}$ from the right we arrive at

$$H_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s) = E_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s) \qquad \text{for } s \in N\,.$$

Moreover, since the eigenvalue problem for $H_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s)$ is equivalent to the eigenvalue problem of $H^U_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s_0)$, cf. [93, Chapter 7], we obtain that for fixed $s \in N$ there exists exactly one point $E_{\rm at}(s) \in \sigma\big(H_{\rm at}(s) \restriction {\rm Ran}P_{\rm at}(s)\big)$, which is in some sense near to $E_{\rm at}(s_0)$. This point is the only eigenvalue of $H_{\rm at}(s)$ in $\big\{z \in \mathbb{C} : |z-E_{\rm at}(s_0)| < \epsilon\big\}$. □

Remark 4.2.25. In case of a non-degenerate $E_{\mathrm{at}}(s_0)$ the existence of an analytic function $s \to E_{\mathrm{at}}(s)$ such that for every s near s_0 the point $E_{\mathrm{at}}(s)$ is near to $E_{\mathrm{at}}(s_0)$ and is an eigenvalue of $H_{\mathrm{at}}(s)$, follows directly from the Kato-Rellich theorem of analytic perturbation theory [115, Theorem XII.8].

Let us now formulate the third and last Hypothesis. In order to do that we define $\overline{P}_{\mathrm{at}}(s) := \mathbb{1}_{\mathcal{H}_{\mathrm{at}}} - P_{\mathrm{at}}(s)$, where $P_{\mathrm{at}}(s)$ is defined in Eq. (4.93). The third Hypothesis is necessary in order to give us a certain control on the resolvent of the non-interacting Hamiltonian $H_0(s)$. More precisely we can deduce from it that $H_0(s)$, for s near s_0, has no eigenvalues belonging to eigenvectors $\psi \in \overline{P}_{\mathrm{at}}(s)\mathcal{H}_{\mathrm{at}} \otimes \mathbb{1}_{\mathcal{F}}$ in a certain neighborhood of the isolated (possibly degenerate) ground-state eigenvalue of $H_{\mathrm{at}}(s_0)$.

Hypothesis III. Hypothesis II holds and there exists a neighborhood $\mathcal{U} \subset X \times \mathbb{C}$ of $(s_0, E_{\mathrm{at}}(s_0))$ such that for all $(s, z) \in \mathcal{U}$, we have $|E_{\mathrm{at}}(s) - z| < 1/2$ and

$$\sup_{(s,z)\in\mathcal{U}} \sup_{q\geq 0} \left\| \frac{q+1}{H_{\mathrm{at}}(s) - z + q} \overline{P}_{\mathrm{at}}(s) \right\| < \infty\,.$$

With the three Hypotheses at hand we can now state the main result of this section.

Theorem 4.2.26. *Suppose Hypotheses I, II and III hold. Then there exists a neighborhood $X_0 \subset X$ of s_0 and a positive constant g_0 such that for all $s \in X_0$ and all $g < g_0$ the operator $H_g(s)$ has an eigenvalue $E_g(s)$ and a corresponding eigenvector $\psi_g(s)$. The eigenvalue and eigenvector are analytic functions of $s \in X_0$ and*

$$E_g(s) = \inf \sigma(H_g(s))$$

for all $s \in X_0 \cap \mathbb{R}^\nu$.

Remark 4.2.27. In case that the irreducibility assumptions of Hypothesis II (ii) is not met the eigenspace of the ground-state eigenvalue is expected to split at higher order in perturbation theory. This phenomenon is known as the Lamb shift and has been considered in the literature [69, 96]. It is natural to assume that the degeneracy of the eigenvalues are lifted until they are protected by a set of symmetries. Hence a combination of an 'n-th'-order split up version of the result in Section 4.1 combined with the result presented in this section would yield a comprehensive answer to analyticity questions for degenerate ground-state eigenvalues in the framework of generalized Spin-Boson models.

In the remainder of Section 4.2 we prove Theorem 4.2.26.

4.2.3 Construction of an effective Hamiltonian

The first step in the operator-theoretic renormalization analysis (cf. Subsection 3.2.2) is to prove that $H_g(s)$ and $H_0(s)$ are a Feshbach pair for a suitable operator $\boldsymbol{\chi}(s)$. Depending on this operator one then constructs an effective Hamiltonian.
We note that the definition of the generalized projection $\boldsymbol{\chi}(s)$ is similar to the related definition of generalized projections in Section 4.1. We choose again smooth functions $\chi, \overline{\chi} \in C^\infty(\mathbb{R}; [0, 1])$ such that $\chi^2 + \overline{\chi}^2 = 1$ and

$$\chi(r) = \begin{cases} 1\,, & \text{if } r \leq \frac{3}{4}\,, \\ 0\,, & \text{if } r \geq 1\,. \end{cases}$$

For $\rho > 0$ we then define

$$\chi_\rho(r) := \chi(r/\rho)\,, \qquad \overline{\chi}_\rho(r) := \overline{\chi}(r/\rho)\,,$$

and set $\chi_\rho := \chi(H_f/\rho)$, $\overline{\chi}_\rho := \overline{\chi}(H_f/\rho)$.
Next we define commuting, non-zero, bounded operators

$$\begin{aligned} \boldsymbol{\chi}_\rho(s) &:= P_{\mathrm{at}}(s) \otimes \chi_\rho\,, \\ \overline{\boldsymbol{\chi}}_\rho(s) &:= \overline{P}_{\mathrm{at}}(s) \otimes \mathbb{1} + P_{\mathrm{at}}(s) \otimes \overline{\chi}_\rho\,. \end{aligned} \tag{4.94}$$

Note that they satisfy $\overline{\boldsymbol{\chi}}_\rho(s)^2 + \boldsymbol{\chi}_\rho(s)^2 = 1$ and that they are in general not self-adjoint. Moreover we set $\boldsymbol{\chi}(s) := \boldsymbol{\chi}_1(s)$ and $\overline{\boldsymbol{\chi}}(s) := \overline{\boldsymbol{\chi}}_1(s)$.

Remark 4.2.28. We note that if Hypothesis II (ii) holds, then also $\boldsymbol{\chi}_\rho$ and $\overline{\boldsymbol{\chi}}_\rho$ commute with $\mathcal{S}$.

The following theorem gives us the conditions for which we can define the first Feshbach operator.

Theorem 4.2.29. *Suppose Hypothesis I, II, III hold, and let $\mathcal{U} \subset X \times \mathbb{C}$ be given by Hypothesis III. Then there is a $g_0 \in \mathbb{R}_+$ such that for all $g \in [0, g_0)$ and all $(s,z) \in \mathcal{U}$, the pair $(H_g(s) - z, H_0(s) - z)$ is a Feshbach pair for $\boldsymbol{\chi}(s)$. Furthermore one has the absolutely convergent expansion*

$$\begin{aligned} &F_{\boldsymbol{\chi}(s)}(H_g(s) - z, H_0(s) - z) \restriction P_{\mathrm{at}}(s) \otimes \mathcal{F} \\ &\quad = E_{\mathrm{at}}(s) - z + H_f + \sum_{L=1}^{\infty} (-1)^{L-1} \boldsymbol{\chi}(s)\, g\, W(s) \Big(\big(H_0(s) - z\big)^{-1} g\, \overline{\boldsymbol{\chi}}(s)\, W(s) \overline{\boldsymbol{\chi}}(s) \Big)^{L-1} \boldsymbol{\chi}(s)\,. \end{aligned} \tag{4.95}$$

In the proof of this theorem we make use the following two lemmas. Since the estimates of these two lemmas do not directly depend on the parameter s, we drop it in the notation. To state the lemmas we define for $R > 0$ and $E \in \mathbb{C}$ the set

$$Q_R(E) := D_R(E) + \mathbb{R}_- \,,$$

where $D_R(E) = \{z \in \mathbb{C} : |E - z| \le R\}$. Moreover we assume that

$$\sup_{z \in Q_R(E)} \|(H_{\mathrm{at}} - z)^{-1} \restriction \mathrm{Ran} \overline{P}_{\mathrm{at}}\| \le \kappa^{-1}. \tag{4.96}$$

Lemma 4.2.30. *Suppose Eq.* (4.96) *holds for $R > 0$ and $E \in \mathbb{C}$. Then for $z \in D_R(E)$ we have*

$$\big\|(H_0 - z)^{-1} \overline{\boldsymbol{\chi}}_\rho\big\| \le (1 + \|P_{\mathrm{at}}\|)\kappa^{-1} + \|P_{\mathrm{at}}\| \big(\tfrac{3}{4}\rho - R\big)^{-1}. \tag{4.97}$$

Proof. We recall that by Definition (4.94) we have $\overline{\boldsymbol{\chi}}_\rho = \overline{P}_{\mathrm{at}} \otimes \mathbb{1} + P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho)$. Clearly H_0 commutes with each of the summands on the right hand side. Let $r \ge 0$ and $z = z' - r$ with $z' \in D_R(E)$, then we have for normalized $\psi \in \mathrm{Ran} P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho)$

$$\big\|(H_0 - z)\psi\big\| \ge |\langle \psi, (H_0 - z)\psi \rangle| \ge \langle \psi, (H_f + r)\psi \rangle - |E - z'| \ge \frac{3}{4}\rho - R\,.$$

This yields

$$\|(H_0 - z)^{-1} \restriction \mathrm{Ran} P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho)\| \le \Big(\frac{3}{4}\rho - R\Big)^{-1}. \tag{4.98}$$

Moreover, we know that the operator H_f is self-adjoint with spectrum $[0, \infty)$, cf. Section 2.2. Hence it follows from the spectral theorem and Eq. (4.96), since $Q_R(E) = D_R(E) + \mathbb{R}_-$, that

$$\sup_{z \in D_R(E)} \big\|(H_0 - z)^{-1} \restriction \mathrm{Ran}\, \overline{P}_{\mathrm{at}} \otimes \mathbb{1}\big\| = \sup_{z \in Q_R(E)} \big\|(H_{\mathrm{at}} - z)^{-1} \restriction \mathrm{Ran}\, \overline{P}_{\mathrm{at}}\big\| \le \kappa^{-1}. \tag{4.99}$$

Now Eq. (4.97) follows using Eqns. (4.98), (4.99) and

$$\|(H_0 - z)^{-1} \overline{\boldsymbol{\chi}}_\rho\| \le \|(H_0 - z)^{-1} \overline{P}_{\mathrm{at}} \otimes \mathbb{1}\| + \|(H_0 - z)^{-1} P_{\mathrm{at}} \otimes \overline{\chi}(H_f/\rho)\|\,.$$

□

Lemma 4.2.31. *Suppose Eq.* (4.96) *holds for $R > 0$ and $E \in \mathbb{C}$. Then for $z \in D_R(E)$ we have*

$$\big\|(H_f + 1)^{1/2} \big(H_0 - z\big)^{-1} \overline{\boldsymbol{\chi}}_\rho\big\| \le 2\, \|\overline{\boldsymbol{\chi}}_\rho\| + (|E_{\mathrm{at}}| + \frac{\rho}{2} + 1) \|\big(H_0 - z\big)^{-1} \overline{\boldsymbol{\chi}}_\rho\| \tag{4.100}$$

Proof. We use that

$$\|(H_f + 1)\psi\| \le 2\, \|H_0\, \psi\| + \|\psi\|\,, \tag{4.101}$$

for all $\psi \in D(H_0)$ and that

$$H_0 (H_0 - z)^{-1} \overline{\boldsymbol{\chi}}_\rho = \overline{\boldsymbol{\chi}}_\rho + z (H_0 - z)^{-1} \overline{\boldsymbol{\chi}}_\rho\,. \tag{4.102}$$

Hence inserting $\psi = (H_0 - z)^{-1} \overline{\boldsymbol{\chi}}_\rho \varphi$ into Eq. (4.101) and using Eq. (4.102) the Estimate (4.100) follows. □

Now we are ready to prove the theorem.

Proof of Theorem 4.2.29. Since Hypotheses I - III hold the parameter s does not play a role in this proof. We always take the supremum over $s \in X$ in all norms without explicitly stating it and suppress the parameter s throughout this proof.

From Hypothesis I we know, that W is a well-defined operators on $\mathcal{H}_{\mathrm{at}} \otimes D(H_f)$ and that it is infinitesimally bounded with respect to H_f. Using in addition Eq. (3.19) from Section 3.3 we obtain the uniform bounds

$$\begin{aligned} \left\| W\,(H_f+1)^{-1/2} \right\| &\le 2\,\|G\|_\mu < \infty\,, \\ \left\| (H_f+1)^{-1/2} W \right\| &\le 2\,\|G\|_\mu < \infty\,. \end{aligned} \tag{4.103}$$

We refer the reader to the proof of Lemma 4.1.7 for more details. On $D(H_0)$ we have by definition

$$\boldsymbol{\chi} H_0 = H_0 \boldsymbol{\chi} \quad \text{and} \quad \overline{\boldsymbol{\chi}} H_0 = H_0 \overline{\boldsymbol{\chi}}\,, \tag{4.104}$$

hence $H_0 - z$ maps $D(H_0) \cap \mathrm{Ran}\overline{\boldsymbol{\chi}}$ into $\mathrm{Ran}\overline{\boldsymbol{\chi}}$, where

$$\mathrm{Ran}\overline{\boldsymbol{\chi}} = \mathrm{Ran}(\overline{P}_{\mathrm{at}} \otimes \mathbb{1}) \oplus \mathrm{Ran}(P_{\mathrm{at}} \otimes \overline{\chi}_1)\,.$$

Moreover, Hypothesis III implies that there exists a $\kappa > 0$ such that Eq. (4.96) holds. Thus we deduce from Lemma 4.2.30 and Lemma 4.2.31 that

$$\sup_{z \in \mathcal{U}} \left\| (H_0 - z)^{-1} \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}} \right\| < \infty\,, \tag{4.105}$$

and

$$\sup_{z \in \mathcal{U}} \left\| (H_f + 1)(H_0 - z)^{-1} \overline{\boldsymbol{\chi}} \right\| < \infty\,. \tag{4.106}$$

Thus as in the proof of Theorem 4.1.6 we obtain that W is infinitesimally bounded with respect to H_0. Therefore H_g is closed on $D(H_0)$. Next we verify the criteria for a Feshbach pair from Lemma 3.2.7. Since Eq. (4.104) is valid on every core of H_0, we get that Condition (a') is satisfied. Moreover H_0 is bounded invertible on $\mathrm{Ran}\overline{\boldsymbol{\chi}}$ (cf. Eq. (4.105)), hence Condition (b') is satisfied. The validity of Condition (c') follows by combining the Eqns. (4.103) and (4.106). More precisely we get that there exits a $g_0 \in \mathbb{R}_+$ such that for all $0 < g < g_0$ the following holds true

$$\begin{aligned} \sup_{z \in \mathcal{U}} \left\| g\, \overline{\boldsymbol{\chi}} W \big(H_0 - z\big)^{-1} \overline{\boldsymbol{\chi}} \right\| &< 1\,, \\ \sup_{z \in \mathcal{U}} \left\| g\, \big(H_0 - z\big)^{-1} \overline{\boldsymbol{\chi}} W \overline{\boldsymbol{\chi}} \right\| &< 1\,. \end{aligned} \tag{4.107}$$

This completes the proof that $(H_g - z, H_0 - z)$ is a Feshbach pair for $\boldsymbol{\chi}$. Now using the definition of the Feshbach map (Eq. (3.4)) and the identity

$$(H_0 - z + g\,\overline{\boldsymbol{\chi}} W \overline{\boldsymbol{\chi}}) \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}} = (H_f+1)^{1/2} A(z) [1 + g\, A(z)^{-1} B(z)] (H_f+1)^{1/2} \restriction \mathrm{Ran}\overline{\boldsymbol{\chi}}$$

where

$$A(z) := (H_f+1)^{-1/2} (H_0 - z)(H_f+1)^{-1/2}\,, \qquad B(z) := (H_f+1)^{-1/2} \overline{\boldsymbol{\chi}} W \overline{\boldsymbol{\chi}} (H_f+1)^{-1/2}\,,$$

we obtain from Neumann's theorem (Theorem 3.2.9) that the expansion (4.95) existence and is absolutely convergent. □

We define the first Feshbach operator using the Neumann expansion of Theorem 4.2.29.

$$H_g^{(1,1)}(s,z) := E_{\mathrm{at}}(s) - z + H_f + \tilde{W}_{\boldsymbol{\chi},g}(s,z)\,, \tag{4.108}$$

where $\tilde{W}_{\boldsymbol{\chi},g}(s,z) \in \mathcal{L}(\mathrm{Ran}P_{\mathrm{at}}(s) \otimes \mathcal{F})$ is given by

$$\tilde{W}_{\boldsymbol{\chi},g}(s,z) = \sum_{L=1}^{\infty} (-1)^{L-1} \boldsymbol{\chi}(s)\, g\, W(s) \Big(\big(H_0(s) - z\big)^{-1} g\, \overline{\boldsymbol{\chi}}(s)\, W(s) \overline{\boldsymbol{\chi}}(s) \Big)^{L-1} \boldsymbol{\chi}(s)\,.$$

Remark 4.2.32. We use the superscript $(1,1)$ in the designation $H_g^{(1,1)}$ to illustrate that we are in the first (initial) step of the renormalization process and that we cut-off photon energies bigger than one. This kind of notation was much more relevant in Section 4.1 where we did two steps of the renormalization analysis by hand and projected onto photon energies strictly smaller than one. Nevertheless we stick with this notation.

Next, using the analytic family $U(s)$ given by Lemma 4.2.20, we define the *effective Hamiltonian* that we consider in this section.

$$\hat{H}_g^{(1,1)}(s,z) := U(s)^{-1} H_g^{(1,1)}(s,z)\, U(s) \restriction \mathrm{Ran} P_{\mathrm{at}}(s_0) \otimes \mathcal{H}_{\mathrm{red}} \,. \tag{4.109}$$

where we denote $U(s) \otimes \mathbb{1}$ by the symbol $U(s)$ and the reduced Fock space $\mathcal{H}_{\mathrm{red}}$ was defined in Eq. (2.12).

Remark 4.2.33. The effective Hamiltonian has the advantages that its domain does not depend on s.

In the following theorem we show that the first Feshbach operator $H_g^{(1,1)}(s,z)$ is analytic on a suitable subset of $X \times \mathbb{C}$. Moreover we show that the effective Hamiltonian (4.109) is isospectral to $H_g(s) - z$. Furthermore the effective Hamiltonian commutes with the set of symmetries $\mathcal{S}$ from Hypothesis II and is reflection symmetric. Note that we make use of the auxiliary operator Q_χ defined in Eq. (3.5) in this theorem.

Theorem 4.2.34. *Suppose Hypothesis I, II, III hold, and let $\mathcal{U} \subset X \times \mathbb{C}$ be given by Hypothesis III. For small enough $g \in \mathbb{R}_+$ we then have that the first Feshbach operator $H_g^{(1,1)}(s,z)$ is analytic on $\mathcal{U}$. Moreover, if $(H_g(s) - z, H_0(s) - z)$ is a Feshbach pair for $\boldsymbol{\chi}(s)$ we have the following*

(a) $H_g(s) - z : D(H_0(s)) \subset \mathcal{H} \to \mathcal{H}$ is bounded invertible if and only if $\hat{H}_g^{(1,1)}(s,z)$ is bounded invertible on $\mathrm{Ran} P_{\mathrm{at}}(s_0) \otimes \mathcal{H}_{\mathrm{red}}$.

(b) The following maps are linear isomorphisms and inverses of each other:

$$U(s)^{-1}\boldsymbol{\chi}(s) : \mathrm{Ker}\,(H_g(s) - z) \to \mathrm{Ker}\, \hat{H}_g^{(1,1)}(s,z)\,,$$
$$Q_{\boldsymbol{\chi}(s)} U(s) : \mathrm{Ker}\, \hat{H}_g^{(1,1)}(s,z) \to \mathrm{Ker}\,(H_g(s) - z)\,.$$

Furthermore, let $\mathcal{S}$ be the set of symmetries given in Hypothesis II, then

(c) $S^* \hat{H}_g^{(1,1)}\, S = \hat{H}_g^{(1,1)}$, for all unitary $S \in \mathcal{S}$.

(d) $S^* \hat{H}_g^{(1,1)}\, S = \hat{H}_g^{(1,1)^*}$, for all antiunitary $S \in \mathcal{S}$.

In addition, if $\overline{X} = X$ we have for $s \in X$ and $z \in \{z \in \mathbb{C} : (s,z) \in \mathcal{U}\}$

(e) $\hat{H}_g^{(1,1)}(s,z)^* = \hat{H}_g^{(1,1)}(\overline{s}, \overline{z})$.

Proof of Theorem 4.2.34. We first prove that $H_g^{(1,1)}(s,z)$ is analytic on $\mathcal{U}$. Using Eq. (4.108) and (4.109) we see that $H_g^{(1,1)}(s,z)$ is analytic, if $W_{\boldsymbol{\chi},g}(s,z)$ is analytic. Now by choosing $|g| < g_0$, we obtain from Theorem 4.2.29 that $W_{\boldsymbol{\chi},g}(s,z)$ is absolutely convergent on $\mathrm{Ran} P_{\mathrm{at}}(s) \otimes \mathcal{F}$ and it remains to prove that the map

$$(s,z) \mapsto \sum_{L=1}^{\infty} (-1)^{L-1} \boldsymbol{\chi}(s)\, g\, W(s) \Big(\big(H_0(s) - z\big)^{-1} g\, \overline{\boldsymbol{\chi}}(s)\, W(s) \overline{\boldsymbol{\chi}}(s) \Big)^{L-1} \boldsymbol{\chi}(s)$$

is analytic in s and z. Since this implies the analyticity of $\langle \alpha, W_{\boldsymbol{\chi},g}(s,z)\beta \rangle$ for all α, $\beta \in \mathrm{Ran} P_{\mathrm{at}}(s) \otimes \mathcal{F}$, which, by Theorem 3.12 of Chapter III in [93], proves that $W_{\boldsymbol{\chi},g}(s,z)$ is analytic in s and z.
Since $\boldsymbol{\chi}(s) W(s)$ and $W(s)\boldsymbol{\chi}(s)$ are analytic, the analyticity of the above map follows if

$$\big(H_0(s) - z\big)^{-1} g\, \overline{\boldsymbol{\chi}}(s)\, W(s) \overline{\boldsymbol{\chi}}(s) \tag{4.110}$$

is analytic. Due to the uniform bound (4.103) one can show that $s \mapsto W(s)(H_f + 1)^{-1/2}$ is analytic on X, cf [66, Lemma 12]. Moreover, from the definition of $\overline{\boldsymbol{\chi}}(s)$ we get the following decomposition

$$\begin{aligned}(H_f + 1)&\big(H_0(s) - z\big)^{-1} \overline{\boldsymbol{\chi}}(s) \\ &= (H_f + 1)\big(H_0(s) - z\big)^{-1} (\overline{P}_{\mathrm{at}}(s) \otimes \mathbb{1}) + (H_f + 1)\big(E_{\mathrm{at}}(s) + H_f - z\big)^{-1} (P_{\mathrm{at}}(s) \otimes \overline{\chi}_1)\,.\end{aligned}$$

In view of Lemma 4.2.24 we can deduce in the same way as in [66, Proposition 27] that the function $(s,z) \mapsto (H_0(s) - z)^{-1}(\overline{P}_{\mathrm{at}}(s) \otimes \mathbb{1})$ is analytic on $\mathcal{U}$. Using a spectral representation of H_f and Hypothesis III it follows that the first factor on the right hand side of the equation above is analytic. The second factor on the right hand side can be viewed as a composition of analytic functions. Hence Eq. (4.110) is analytic. This concludes the proof that $H_g^{(1,1)}(s,z)$ is analytic on $\mathcal{U}$.

The Statements (a) and (b) follow in view of Remark 4.2.33 from Theorem 3.2.4 by making the choice $V = \mathrm{Ran}\big(P_{\mathrm{at}}(s_0) \otimes P_{[0,1]}(H_f)\big)$.
Statements (c) and (d) follow from Lemma 4.2.20, the definitions of the terms in $\hat{H}_g^{(1,1)}$, see Eqns. (4.108) and (4.109), and the symmetry properties of all the operators occuring in the definitions. In particular see Remark 4.2.4 and 4.2.6. Thus it remains to prove (e).
For this let $s \in X \cap \mathbb{R}^d \neq \emptyset$ and $z \in \{z \in \mathbb{C} : (s,z) \in \mathcal{U}\}$. Since the analytic family $U(s)$ is unitary for real s and Hypothesis I and II hold we obtain

$$\begin{aligned}
\hat{H}_g^{(1,1)}(s,z)^* &= \Big(U(s)^{-1} H_g^{(1,1)}(s,z)\, U(s) \restriction \mathrm{Ran} P_{\mathrm{at}}(s_0) \otimes \mathcal{H}_{\mathrm{red}}\Big)^* \\
&= (U(s))^* \, (H_g^{(1,1)}(s,z))^* \, \big(U(s)^{-1}\big)^* \restriction \mathrm{Ran} P_{\mathrm{at}}(s_0) \otimes \mathcal{H}_{\mathrm{red}} \\
&= U(s)^{-1} H_g^{(1,1)}(s,\overline{z})\, U(s) \restriction \mathrm{Ran} P_{\mathrm{at}}(s_0) \otimes \mathcal{H}_{\mathrm{red}} \\
&= \hat{H}_g^{(1,1)}(s,\overline{z})\,.
\end{aligned}$$

Additionally, since $\overline{X} = X$, we get that $H_g^{(1,1)}(\overline{s},z)^*$ is also analytic in $\mathcal{U}$ and coincides with $H_g^{(1,1)}(s,\overline{z})$ for $s \in \mathbb{R}^\nu \cap X$. We therefore get by the unique continuation property of analytic functions (cf. [86]) that

$$H_g^{(1,1)}(s,z)^* = H_g^{(1,1)}(\overline{s},\overline{z})\,,$$

for all $s \in X$ and $z \in \{z \in \mathbb{C} \,|\, (s,z) \in \mathcal{U}\}$. This completes the proof of Theorem 4.2.34. □

Remark 4.2.35. Let $g \in (0, g_0)$ and $U_0(s) := \{z \in \mathbb{C} : (s,z) \in \mathcal{U}\}$ be a neighborhood of $E_{\mathrm{at}}(s)$. Then Theorem 4.2.34 implies that finding an eigenvalue of $H_g(s)$ in $U_0(s)$ is the same as finding $z \in \mathbb{C}$ such that $\hat{H}_g^{(1,1)}(s,z)$ has non-trivial kernel. To see this recall the isospectrality property of the smooth Feshbach map (Eq. (3.6)).

4.2.4 The renormalization transformation

In this subsection we define the renormalization transformation $\mathcal{R}_\rho$ and study important properties of this map. Moreover we show that the effective Hamiltonian $\hat{H}_g^{(1,1)}$ lies in the domain of the renormalization transformation. But to begin with we state an abstract existence result for Feshbach pairs. For this recall from Section 3.3 that the map $H : \mathcal{W}_\xi^{[d]} \to \mathcal{L}(\mathcal{H}_{\mathrm{red}})$ for $w = (w_{m,n})_{m,n \in \mathbb{N}_0} \in \mathcal{W}_\xi^{[d]}$ is given by

$$H(w) := \sum_{m,n} H_{m,n}(w_{m,n})\,,$$

where for $m + n \geq 1$ the operator $H_{m,n}$ is defined by Eq. (3.10) and $H_{0,0}(w) := w_{0,0}(H_f)$. Moreover we note that $w_{0,0}(0) = \langle \Omega, w_{0,0}(H_f)\, \Omega\rangle = \langle \Omega, H(w)\Omega\rangle =: \langle H(w)\rangle_\Omega$.
As in Section 4.1 we define a neighborhood of the free field Hamiltonian in terms of the norms introduced in Subsection 3.3.1 as follows. For $\alpha, \beta, \gamma \in \mathbb{R}_+$ let $\mathcal{B}^{[d]}(\alpha,\beta,\gamma) \subset H(\mathcal{W}_\xi^{[d]})$ be defined by

$$\mathcal{B}^{[d]}(\alpha,\beta,\gamma) := \Big\{ H(w) : \|w_{0,0}(0)\| \leq \alpha, \|w'_{0,0} - 1\|_\infty \leq \beta, \|w - w_{0,0}\|^{\#}_{\mu,\xi} \leq \gamma \Big\}\,.$$

Lemma 4.2.36. *Suppose $\rho,\ \xi \in (0,1)$ and $\mu > 0$. If $H(w) \in \mathcal{B}^{[d]}(\rho/2, \rho/8, \rho/8)$, then $(H(w), H_{0,0}(w))$ is a Feshbach pair for χ_ρ.*

This lemma follows up to notational differences directly from the proof of Lemma 15 in [66]. Moreover a similar proof is given in [57]. Therefore we omit the proof of this lemma.

Definition 4.2.37. The so-called *renormalization transformation*

$$\mathcal{R}_\rho := S_\rho \circ F_{\chi_\rho} \tag{4.111}$$

is defined on the domain $D(F_{\chi_\rho})$ of the Feshbach map F_{χ_ρ}. Note that the map S_ρ, which is called rescaling by dilation, was defined in Eq. (3.14).

Now let $\big(H(w), H_{0,0}(w)\big)$ be a Feshbach pair with respect to χ_ρ. Then we can deduce from Theorem 3.3.4 that the map $\mathcal{R}_\rho$ has a well-defined domain $D(\mathcal{R}_\rho) \subset \mathcal{L}(\mathcal{H}_{\mathrm{red}})$. Moreover the renormalization transformation is in this case explicitly given by the following bounded operator on $\mathcal{H}_{\mathrm{red}}$,

$$\mathcal{R}_\rho(H(w)) = \rho^{-1} \Gamma_\rho F_{\chi_\rho}(H(w), H_{0,0}(w)) \Gamma_\rho^*\,.$$

Furthermore we can derive more properties of $\mathcal{R}_\rho$. Namely, from Lemma 4.2.36 we can deduce that the neighborhood $\mathcal{B}^{[d]}(\rho/2, \rho/8, \rho/8)$ is contained in $D(\mathcal{R}_\rho)$. In addition, Theorem 3.2.4 implies that $\operatorname{Ker} \mathcal{R}_\rho(H(w))$ is isomorphic to $\operatorname{Ker} H(w)$. The subsequent theorem shows that we even have some control on the range of the renormalization transformation.

Theorem 4.2.38. *There exists a constant $C_\chi \geq 1$ that only depends on χ such that for $\mu > 0$, $\rho \in (0,1)$, $\xi := \frac{\sqrt{\rho}}{4C_\chi}$ and $\beta, \gamma \leq \frac{\rho}{8C_\chi}$, we have*

$$\mathcal{R}_\rho - \rho^{-1}\langle \cdot \rangle_\Omega : \mathcal{B}^{[d]}(\rho/2, \beta, \gamma) \to \mathcal{B}^{[d]}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}),$$

where

$$\tilde{\alpha} = 3\, C_\chi \frac{\gamma^2}{2\rho}, \quad \tilde{\beta} = \beta + 3\, C_\chi \frac{\gamma^2}{2\rho}, \quad \tilde{\gamma} = 128\, C_\chi^2 \rho^\mu \gamma .$$

This theorem is a variant of Theorem 16 in [66], which is itself a variant of Theorem 3.8 in [14]. For a proof we refer to [14] and remark that our definition of the renormalization transformation is different to the one they used. To be more precise, we do not use an analytic deformation of the spectral parameter. Moreover we consider matrix-valued integral kernels but due to our definition of the related Banach spaces (Section 3.3) there are no actual differences in the proof, expect minor notational ones. Furthermore we refer the reader to [57, Appendix 1] for a detailed derivation of the constants $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. From that proof we additionally conclude that without loss of generality we can set

$$C_\chi := \frac{4}{3}\left(\|\chi_1\|_\infty + \|\chi_1'\|_\infty + \|\chi_1'\|_\infty^2\right). \tag{4.112}$$

For later use we additionally define the following constants

$$C_\beta := \frac{3}{2}\, C_\chi , \qquad C_\gamma := 256\, C_\chi^2 . \tag{4.113}$$

In the next subsection we iteratively apply the renormalization transformation $\mathcal{R}_\rho$ to the initially defined isospectral operator $\hat{H}_g^{(1,1)}$. Hence we need to show that we are actually allowed to do that, i.e. we have to prove that $\hat{H}_g^{(1,1)}(s,z) - \langle \hat{H}_g^{(1,1)}(s,z)\rangle_\Omega \in D(\mathcal{R}_\rho)$.
In the following we denote by

$$d_0 := \dim\left(\operatorname{Ran} P_{\mathrm{at}}(s_0)\right) = \dim\left(\operatorname{Ran} P_{\mathrm{at}}(s)\right) \tag{4.114}$$

the dimension of the eigenspace corresponding to the ground-state eigenvalue of $H_{\mathrm{at}}(s)$.

Theorem 4.2.39. *Suppose the Hypotheses I, II and III are true for some $\mu > 0$ and $\mathcal{U} \subset \mathbb{C}^\nu \times \mathbb{C}$. Then, for all $\xi \in (0,1)$ and arbitrarily positive constants α_0, β_0 and γ_0, there exits a positive constant g_1 such that for all $g \in [0, g_1)$ and all $(s,z) \in \mathcal{U}$, $(H_g(s) - z, H_0(s) - z)$ is a Feshbach pair for $\boldsymbol{\chi}(s)$, and*

$$\hat{H}_g^{(1,1)}(s,z) - (E_{\mathrm{at}}(s) - z) \in \mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0).$$

Using Theorem 4.2.29 we directly obtain that the Feshbach property is satisfied for sufficiently small g. Hence to prove the theorem it remains to construct a sequence of integral kernels $w \in \mathcal{W}_\xi^{[d_0]}$ such that $\hat{H}_g^{(1,1)}(s,z) = H(w)$. Due to the definition of the space $\mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0)$, the assumptions of Hypothesis I, II, III and the definition of $\hat{H}_g^{(1,1)}$ this construction is similar to the one in [66] where a sequence of integral kernels with values in $C^1([0,1])$ was constructed. In addition we refer the reader to Subsection 4.1.3 in Section 4.1. There we constructed a sequence of matrix-valued integral kernels in a similar situation. Hence we do not repeat this construction here.

Remark 4.2.40. The set $\mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0)$ is in general not a subset of $D(\mathcal{R}_\rho)$. However in Subsection 4.2.6 we show that we can make appropriate choices for ξ, α_0, β_0 and γ_0 such that $\mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0) \subset D(\mathcal{R}_\rho)$.

4.2.5 Iterating the renormalization transformation $\mathcal{R}_\rho$

In this subsection we repeatedly apply the renormalization transformation $\mathcal{R}_\rho$ to the operator $\hat{H}_g^{(1,1)}$. To do that we first study properties that are preserved by the renormalization map. In particular we show that analyticity and symmetry properties are preserved under renormalization. These results provide us with key tools to make the operator-theoretic renormalization method applicable in the degenerate case. Then we construct, with help of the renormalization transformation, a sequence of non-empty, shrinking sets $U_n(s) \searrow \{z_\infty(s)\}$ such that the limiting point $z_\infty(s)$ is an eigenvalue of the operator $H_g(s)$. We want to mention that for this construction we make use of the irreducibility assumption from Hypothesis II.

Renormalization preserves analyticity and symmetry

In [66, Proposition 17], Griesemer and Hasler proved that analyticity is preserved under renormalization. The following proposition is a copy of their result, with the obvious modifications in notation.

Proposition 4.2.41 (Proposition 17, [66]). *Let X be an open subset of $\mathbb{C}^{\nu+1}$ with $\nu \geq 0$. Suppose that the map $\sigma \mapsto H(w^\sigma) \in \mathcal{L}(\mathcal{H}_{\mathrm{red}})$ is analytic on X, and that $H(w^\sigma)$ belongs to some neighborhood $\mathcal{B}^{[d]}(\alpha,\beta,\gamma)$ for all $\sigma \in X$. Then*

(a) $H_{0,0}(w^\sigma)$ is analytic on X.

(b) If for all $\sigma \in X$, $(H(w^\sigma), H_{0,0}(w^\sigma))$ is a Feshbach pair for χ_ρ, then $F_{\chi_\rho}(H(w^\sigma), H_{0,0}(w^\sigma))$ is analytic on X.

Roughly speaking a direct consequence of this proposition is the following. If we start with a properly chosen analytic operator in the domain of the renormalization transformation $\mathcal{R}_\rho$ and apply the transformation n times, then the resulting operator is analytic as well. This is, alongside domain issues, the main ingredient to the proof of the analyticity in Theorem 4.2.26 in Subsection 4.2.6. In the same spirit, we obtain that the following properties (cf. Definition 4.2.17) are preserved under renormalization.

Proposition 4.2.42. *Let X be an open subset of $\mathbb{C}^{\nu+1}$ with $X = \overline{X}$, $\nu \geq 0$. Let $\mathcal{S}$ be a set of symmetries acting on $\mathcal{H}_{\mathrm{red}}$ and commuting with H_f and S_ρ. Suppose $\sigma \mapsto H(w^\sigma) \in \mathcal{L}(\mathcal{H}_{\mathrm{red}})$ is analytic on X, reflection symmetric and commutes with the set of symmetries $\mathcal{S}$. Moreover assume that $H(w^\sigma)$ belongs to some neighborhood $\mathcal{B}^{[d]}(\alpha,\beta,\gamma)$ for all $\sigma \in X$. Then*

(a) $H_{0,0}(w^\sigma)$ is reflection symmetric and commutes with $\mathcal{S}$.

(b) If for all $\sigma \in X$, $(H(w^\sigma), H_{0,0}(w^\sigma))$ is a Feshbach pair for χ_ρ, then $F_{\chi_\rho}(H(w^\sigma), H_{0,0}(w^\sigma))$ is reflection symmetric and commutes with $\mathcal{S}$. Moreover the same holds true for $\mathcal{R}_\rho(H(w^\sigma))$.

Proof. $H_{0,0}(w^\sigma)$ is analytic by Proposition 4.2.41 and, if for all $\sigma \in X$, $(H(w^\sigma), H_{0,0}(w^\sigma))$ is a Feshbach pair for χ_ρ, then also $F_{\chi_\rho}(H(w^\sigma), H_{0,0}(w^\sigma))$ is analytic. Moreover the map $w^\sigma \mapsto H(w^\sigma)$ is injective by Theorem 3.3.4. Hence there exists a unique decomposition

$$H(w^\sigma) = \sum_{m,n} H_{m,n}(w^\sigma)\,.$$

Now let $W := H(w^\sigma) - H_{0,0}(w^\sigma)$. Then it follows that $H_{0,0}(w^\sigma)$ and

$$F_{\chi_\rho}\big(H(w^\sigma), H_{0,0}(w^\sigma)\big) = H_{0,0}(w^\sigma) + \chi_\rho W \chi_\rho - \chi_\rho W \overline{\chi_\rho}\big(H_{0,0}(w^\sigma) + \overline{\chi_\rho}\, W\, \overline{\chi_\rho}\big)^{-1} \overline{\chi_\rho}\, W \chi_\rho\,,$$

are reflection symmetric and commute with $\mathcal{S}$. In addition we directly get from this and the assumptions on the set $\mathcal{S}$ that $\mathcal{R}_\rho = S_\rho \circ F_{\chi_\rho}$ has the same properties. □

Remark 4.2.43. This *symmetry preservation property* of the renormalization transformation $\mathcal{R}_\rho$ plays an essential role for the construction of the eigenvalue $z_\infty(s)$ of the operator $H_g(s)$ in the next part of this subsection. More precisely it simplifies that construction because we do not need to worry about a split up of the corresponding eigenspace.

Iterating the renormalization transformation $\mathcal{R}_\rho$

In the following we construct a sequence of non-empty, shrinking sets such that the limiting point is an eigenvalue of $H_g(s)$. For this propose we define the following iterative sequence

$$H_g^{[n]}[s,z] := \mathcal{R}_\rho^n\big(H_g^{[0]}[s,z]\big) \tag{4.115}$$

of operators on $\mathcal{H}_{\mathrm{red}}$. This sequence is, by Theorem 3.2.4, isospectral in the sense, that $\operatorname{Ker} H_g^{[n+1]}[s,z]$ is isomorphic to $\operatorname{Ker} H_g^{[n]}[s,z]$.

Moreover, we mentioned in Remark 4.2.35, that finding an eigenvalue of the operator $H_g(s)$ in the neighborhood $U_0(s) := \{z \in \mathbb{C} : (s,z) \in \mathcal{U}\}$ of $E_{\mathrm{at}}(s)$ is, for small $|g|$, the same as finding $z \in \mathbb{C}$ such that $\hat{H}_g^{(1,1)}(s,z)$ has non-trivial kernel. In Subsection 4.2.6 we will choose $H_g^{[0]}[s,z] = \hat{H}_g^{(1,1)}(s,z)$. Therefore, if we find a point $z_\infty(s)$ such that $H_g^{[n]}(s, z_\infty(s))$ has non-trivial kernel for $n \to \infty$, then this point is an eigenvalue of $H_g(s)$.

We begin the search for such a point by proving that the sequence of operators in Eq. (4.115) is well-defined for all $z \in U_n(s)$, where the sets $U_n(s)$ are non-empty and shrinking sets for $n \to \infty$.

Remark 4.2.44. The following construction of the sets $U_n(s)$ is similar to the corresponding construction in [66, Chapter 8]. Except that our neighborhoods of the free field energy are defined for matrix-valued integral kernels and the operator $H_g^{[0]}[s,z]$ is reflection symmetric and commutes with a set of symmetries $\mathcal{S}$. We make use of these properties during the construction to show that the vacuum expectation value of $H_g^{[n]}[s,z]$ is a d-dimensional diagonal matrix with equal entries, cf. Eq. (4.116) below.
Our exact assumptions are as follows

(A) $U_0(s)$ is an open subset of $\mathbb{C}$ and, for every $z \in U_0(s)$,

$$H_g^{[0]}[s,z] \in \mathcal{B}^{[d]}(\infty, \rho/8, \rho/8)$$

is reflection symmetric and commutes with a set of symmetries $\mathcal{S}$ as in Hypothesis II. Moreover, for $\mu > 0$ the neighborhood $\mathcal{B}^{[d]}(\infty, \rho/8, \rho/8) \subset H(\mathcal{W}_\xi^{[d]})$ is given in terms of $\xi := \frac{\sqrt{\rho}}{(4C_\chi)}$, where $\rho \in (0,1)$ and C_χ is defined by Eq. (4.112).

Remark 4.2.45. Using Theorem 4.2.34 and Theorem 4.2.39 we conclude that for the model defined in Subsection 4.2.2 and in the case that Hypotheses I, II and III hold, it is possible to choose an open set $U_0(s)$ and appropriate constants μ and ξ such that Assumption (A) holds. We elaborate on that in the proof of the main theorem (Subsection 4.2.6).

From now on to the end of this subsection we omit the parameters s and g since all appearing estimates are uniform in $s \in X$ and $|g| < g_0$ for some $g_0 > 0$.
In order to begin our construction of the set U_n we recall that Lemma 4.2.36 and Proposition 4.2.42 imply the following. The recursively defined operator

$$H^{[N]}[z] = \mathcal{R}_\rho\big(H^{[N-1]}[z]\big)$$

is, for every $N \geq 1$, well-defined, reflection symmetric and commutes with a set of symmetries $\mathcal{S}$ if the operators $H^{[0]}[z], \ldots, H^{[N-1]}[z]$ are reflection symmetric, commute with the same set of symmetries $\mathcal{S}$ and belong to the neighborhood $\mathcal{B}^{[d]}(\rho/2, \rho/8, \rho/8)$. Using Assumption (A) we see that Theorem 4.2.38 gives us sufficient condition for this to happen.
Namely, starting with initial values β_0, $\gamma_0 \in \mathbb{R}_+$ we deduce from Theorem 4.2.38 the following iterated values for $n = 1, \ldots, N-1$

$$\gamma_n := (C_\gamma \rho^\mu)^n \gamma_0\,, \qquad \text{and} \qquad \beta_n := \beta_0 + \left(\frac{C_\beta}{\rho} \sum_{k=0}^{n-1} (C_\gamma \rho^\mu)^{2k}\right) \gamma_0^2\,.$$

Moreover, if $C_\gamma \rho^\mu < 1$ and β_0, γ_0 are sufficiently small, the following inequalities are satisfied

$$\gamma_n \leq \frac{\rho}{8C_\chi}\,, \qquad \beta_n \leq \frac{\rho}{8C_\chi}\,.$$

Let us assume that this is the case for all $n < N$. Then, due to Assumption (A) and the symmetry preservation property of $\mathcal{R}_\rho$, we can deduce from Lemma 4.2.19 that

$$E^{(n)}(z) := \langle \Omega, H^{[n]}[z]\, \Omega \rangle\,, \tag{4.116}$$

is a d-dimensional diagonal matrix with all entries equal to $f^{(n)}(z)$ for some function $f^{(n)} : \mathcal{U} \to \mathbb{C}$. We note that it is crucial for the iteration of the renormalization transformation that E^n is a multiple of the identity. If we can guarantee that

$$\|E^{(n)}(z)\| = |f^{(n)}(z)| \leq \frac{\rho}{2}\,,$$

for every $n = 0, \ldots, N-1$, then the operator $H^{[N]}[z]$ is well-defined, reflection symmetric and commutes with the set of symmetries $\mathcal{S}$. We achieve this by restricting the permissible values of z in every step and define recursively for all $n \geq 1$

$$U_n := \{ z \in U_{n-1} : |f^{(n-1)}(z)| \leq \rho/2 \}\,. \tag{4.117}$$

Hence for $z \in U_N$ we can apply Theorem 4.2.38 again and get $H^{[N]}[z] \in \mathcal{B}^{[d]}(\infty, \rho/8, \rho/8)$ and

$$\left| \big|f^{(N)}(z)\big| - \frac{\big|f^{(N-1)}(z)\big|}{\rho} \right| \leq \frac{C_\beta}{\rho} \gamma_{N-1}^2 =: \alpha_N\,.$$

The subsequent lemma is a summary of the above construction.

Lemma 4.2.46. *Suppose that Assumption (A) holds with $\rho \in (0,1)$ sufficiently small such that $C_\gamma \rho^\mu < 1$. Moreover suppose $\beta_0, \gamma_0 \le \frac{\rho}{8C_\chi}$ and*

$$\beta_0 + \frac{C_\beta\, \rho^{-1}}{1-(C_\gamma \rho^\mu)^2}\, \gamma_0^2 \le \frac{\rho}{8C_\chi}. \tag{4.118}$$

Then for all $n \in \mathbb{N}$ the following is satisfied. The operator $H^{[n]}[z]$ is well-defined, reflection symmetric and commutes with the set of symmetries $\mathcal{S}$ for all $z \in U_n$, if $H^{[0]}[z] \in \mathcal{B}^{[d]}(\infty, \beta_0, \gamma_0)$ for all $z \in U_0$.
In addition,

$$H^{[n]}[z] - \frac{1}{\rho} E^{(n-1)}(z) \in \mathcal{B}^{[d]}(\alpha_n, \beta_n, \gamma_n)\,, \tag{4.119}$$

with

$$\begin{aligned}
\alpha_n &:= \frac{C_\beta}{\rho}\gamma_{n-1}^2\,,\\
\beta_n &:= \beta_0 + \left(\frac{C_\beta}{\rho}\sum_{k=0}^{n-1}(C_\gamma\rho^\mu)^{2k}\right)\gamma_0^2\,,\\
\gamma_n &:= (C_\gamma\rho^\mu)^n\gamma_0\,.
\end{aligned}$$

Next we show that the sets U_n are non-empty for all $n \in \mathbb{N}$. In order to do that we introduce the discs

$$D_r := \{z \in \mathbb{C} : |z| \le r\}$$

and note that by definition $U_n = {f^{(n-1)}}^{-1}(D_{\rho/2})$ for $n \ge 1$. Moreover we denote by $B(E,\rho) \subset \mathbb{C}$ the open ball with radius $\rho > 0$ around $E \in \mathbb{C}$.

Definition 4.2.47. We call a function $f : A \to B$ *conformal* if it is the restriction of an analytic bijection $f : U \to V$ between open sets $U \supset A$, $V \supset B$ and if $f(A) = B$.

The statement of the following theorem is used in the proof of the subsequent lemma. More precisely we employ it to show that the map $f^{(n)} : U_n \to \mathbb{C}$ is conformal.

Theorem 4.2.48 (symmetric Rouché's theorem, [45])**.** *Let $K \subset G$ be a bounded region with continuous boundary ∂K. Moreover, let f and g denote holomorphic functions on G. Then f and g have the same number of roots in K if the strict inequality*

$$|f(z) - g(z)| < |f(z)| + |g(z)|\,, \qquad z \in \partial K\,,$$

holds on the boundary ∂K.

Proof. [45]; Let $C\colon [0,1] \to \mathbb{C}$ be a simple closed curve whose image is the boundary ∂K. The hypothesis implies that f has no roots on ∂K, hence by the argument principle, the number $N_f(K)$ of zeros of f in K is

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{f\circ C}\frac{dz}{z} =: \operatorname{Ind}_{f\circ C}(0)\,,$$

i.e., the winding number of the closed curve $f \circ C$ around the origin; similarly for g. The hypothesis ensures that $g(z)$ is not a negative real multiple of $f(z)$ for any $z = C(x)$. Thus 0 does not lie on the line segment joining $f(C(x))$ to $g(C(x))$, and

$$H_t(x) = (1-t)f(C(x)) + tg(C(x))$$

is a homotopy between the curves $f \circ C$ and $g \circ C$ avoiding the origin. The winding number is homotopy-invariant [120, Lemma 2.1]. Therefore the function

$$I(t) := \operatorname{Ind}_{H_t}(0) = \frac{1}{2\pi i}\oint_{H_t}\frac{dz}{z}$$

is continuous and integer-valued, hence constant. This shows

$$N_f(K) = \operatorname{Ind}_{f\circ C}(0) = \operatorname{Ind}_{g\circ C}(0) = N_g(K)\,. \qquad \square$$

Remark 4.2.49. The original Rouché's theorem follows from Theorem 4.2.48 by setting $f(z) := f(z)+g(z)$ and $g(z) := f(z)$.

The following lemma shows that the sets U_n are non-empty and that there exists a limiting point for $n \to \infty$. It is a combination of Lemma 19, Corollary 20 and Proposition 21 in [66], adjusted to our problem and notation. The proof is based on the corresponding proofs in [66] with some modifications. For the convenience of the reader we give a complete proof nevertheless.

Lemma 4.2.50. *Suppose that Assumption (A) holds with $E_{\mathrm{at}} \in U_0$ and $\rho \in (0, 4/5)$ sufficiently small such that $C_\gamma \rho^\mu < 1$ and $\overline{B(E_{\mathrm{at}}, \rho)} \subset U_0$. Suppose further that $\alpha_0 < \rho/2$, $\beta_0, \gamma_0 \leq \rho/(8C_\chi)$, and that Ineq. (4.118) holds. If $z \mapsto H^{[0]}[z] \in \mathcal{L}(\mathcal{H}_{\mathrm{at}})$ is analytic in U_0 and*

$$H^{[0]}[z] - (E_{\mathrm{at}} - z) \in \mathcal{B}^{[d]}(\alpha_0, \beta_0, \gamma_0)$$

for all $z \in U_0$, then the following assertions are true.

(a) For $n \geq 0$, the map $E^{(n)} : U_n \to \mathcal{L}(\mathbb{C}^d)$ is a analytic in U_n°. The d-times repeated diagonal entry $f^{(n)}$ of the matrix $E^{(n)}$ is a conformal map from U_{n+1} onto $D_{\rho/2}$. Moreover there exists a unique zero z_n for $E^{(n)}$ or rather $f^{(n)}$ in the set U_n. Additionally we have the inclusions

$$B(E_{\mathrm{at}}, \rho) \supset U_1 \supset U_2 \supset U_3 \supset \dots .$$

(b) The limit $z_\infty := \lim_{n\to\infty} z_n$ exists and for $\epsilon := 1/2 - \rho/2 - \alpha_1 > 0$, we have

$$|z_n - z_\infty| \leq \rho^n \exp\left(\frac{1}{2\rho\epsilon^2} \sum_{k=0}^{\infty} \alpha_k\right).$$

(c) Let $E_{\mathrm{at}} \in \mathbb{R}$ and $H^{[0]}[z]^* = H^{[0]}[\overline{z}]$ for all $z \in \overline{B(E_{\mathrm{at}}, \rho)}$. Then there exists an $a < z_\infty$ such that $H^{[0]}[x]$ has a bounded inverse for all $x \in (a, z_\infty)$.

Proof. (a): The map $z \mapsto H^{[0]}[z]$ is analytic for $z \in U_0$. Hence Proposition 4.2.41 implies that $z \mapsto H^{[n]}[z]$ is analytic for $z \in U_n^\circ$ for all $n \in \mathbb{N}$. By definition it follows that the map $E^{(n)} : U_n \to \mathcal{L}(\mathbb{C}^d)$ is analytic in U_n° as well. Moreover by Lemma 4.2.19 and the symmetry preservation property of $\mathcal{R}_\rho$ we obtain that $E^{(n)}$ is a d-dimensional diagonal matrix with entries $f^{(n)} : U_n \to \mathbb{C}$ for all $n \in \mathbb{N}_0$.
Next we use induction to show that $f^{(n)}$ is a conformal mapping from U_{n+1} to $D_{\rho/2}$. Obviously $D_{\rho/2} \subset \mathbb{C}$ and $f^{(0)}$ is analytic on U_0°. Additionally we have by assumption that $\overline{B(E_{\mathrm{at}}, \rho)} \subset U_0$ and

$$\|E^{(0)}(z) - (E_{\mathrm{at}} - z)\| \leq \alpha_0 \quad \text{for all } z \in U_0. \tag{4.120}$$

This implies for $\epsilon > 0$ and $z \in {f^{(0)}}^{-1}(D^\circ_{\rho/2+\epsilon})$ that

$$|E_{\mathrm{at}} - z| \leq \alpha_0 + \frac{\rho}{2} + \epsilon < \rho\,,$$

where the last inequality only holds true for sufficiently small ϵ because $\alpha_0 < \rho/2$ by assumption. Therefore

$$U_1 \subset {f^{(0)}}^{-1}(D^\circ_{\rho/2+\epsilon}) \subset B(E_{\mathrm{at}}, \rho) \subset U_0.$$

Moreover, ${f^{(0)}}^{-1}(D^\circ_{\rho/2+\epsilon})$ is an open set in $\mathbb{C}$ since $f^{(0)}$ is continuous on U_0°. Hence $f^{(0)}$ is conformal from U_1 onto $D_{\rho/2}$ if we prove that it is a bijection from ${f^{(0)}}^{-1}(D^\circ_{\rho/2+\epsilon})$ to $D^\circ_{\rho/2+\epsilon}$. In order to do this we use Theorem 4.2.48 (symmetric Rouché's theorem). For that we choose an arbitrary $w \in D^\circ_{\rho/2+\epsilon}$. Then there exists exactly one $z \in B(E_{\mathrm{at}}, \rho)$ such that

$$E_{\mathrm{at}} - z - w = 0\,,$$

and for all elements of the boundary, $z \in \partial B(E_{\mathrm{at}}, \rho)$, we get

$$|E_{\mathrm{at}} - z - w| \geq |E_{\mathrm{at}} - z| - |w| \geq \frac{\rho}{2} > \alpha_0\,.$$

Now, due to Eq. (4.120), we have the inequality

$$\left|(f^{(0)}(z) - w) - (E_{\mathrm{at}} - z - w)\right| \leq \alpha_0$$

for all $z \in \overline{B(E_{\mathrm{at}},\rho)}$. Thus from the symmetric Rouché theorem it follows that there exists exactly one $z \in \overline{B(E_{\mathrm{at}},\rho)}$ such that

$$f^{(0)}(z) - w = 0\,.$$

Since $f^{(0)}$ is by construction surjective and ${f^{(0)}}^{-1}(D^\circ_{\rho/2+\epsilon}) \subset D(E_{\mathrm{at}},\rho)$ we have proven bijectivity. Thus we have shown so far

$$U_1 \subset B(E_{\mathrm{at}},\rho) \text{ and } f^{(0)} : U_1 \to D_{\rho/2} \text{ is conformal.} \tag{4.121}$$

We use this as the basis for our induction. More precisely, we prove by induction that

$$(\mathbf{I}_n): \qquad f^{(n-1)} : U_n \to D_{\rho/2} \text{ is conformal.}$$

For $n = 1$ this follows from Eq. (4.121). Now suppose $(\mathbf{I}_n)$ holds. We know that $C_\chi > 1$ and $\rho < 4/5$. Moreover, since $C_\gamma \rho^\mu < 1$, we also know that $\alpha_n \leq \alpha_1 = (C_\beta/\rho)\gamma_0^2$. Hence Inequality (4.118) implies that $\alpha_n \leq \rho/8$. Combining this with $\rho/2 < 4/10$ we get $\alpha_n + \rho/2 < 1/2$, and thus we can choose $\epsilon > 0$ such that

$$\alpha_n + \frac{\rho}{2} + 2\epsilon < \frac{1}{2}\,. \tag{4.122}$$

Now we define $D^\circ_+ := D^\circ_{\rho/2+\epsilon}$ and $D^\circ_- := D^\circ_{\rho/2-\rho\epsilon}$, then clearly we have $D^\circ_- \subset D_{\rho/2} \subset D^\circ_+$. Furthermore we have $|f^{(n)}(z) - \rho^{-1} f^{(n-1)}(z)| \leq \alpha_n$, due to Lemma 4.2.46 (Eq. (4.119)). In addition, if we choose $z \in {f^{(n)}}^{-1}(D^\circ_+)$, i.e. $|f^{(n)}(z)| < \rho/2 + \epsilon$, then Eq. (4.122) implies $|f^{(n-1)}(z)| < \rho/2 - \rho\epsilon$ and therefore

$${f^{(n)}}^{-1}(D^\circ_+) \subset {f^{(n-1)}}^{-1}(D^\circ_-)\,. \tag{4.123}$$

Applying the induction hypothesis $(\mathbf{I}_n)$ we deduce that ${f^{(n)}}^{-1}(D^\circ_+) \subset U^\circ_n$. As before the inverse map ${f^{(n)}}^{-1}(D^\circ_+)$ is open because $f^{(n)}$ is continuous. Thus it remains to prove that $f^{(n)} :\ {f^{(n)}}^{-1}(D^\circ_+) \to D^\circ_+$ is a bijection, since this and the analyticity of $f^{(n)}$ then imply $(\mathbf{I}_{n+1})$. We use again the symmetric Rouché's theorem to prove this. For this let $w \in D^\circ_+$, then we have $\rho w \in D^\circ_-$. Moreover the induction hypothesis $(\mathbf{I}_n)$ implies that there exists exactly one $z \in {f^{(n-1)}}^{-1}(D^\circ_-)$ such that

$$f^{(n-1)}(z) - \rho w = 0\,.$$

For all $z \in \partial\big({f^{(n-1)}}^{-1}(D^\circ_-)\big)$ we deduce from Eq. (4.122) that

$$\big|\rho^{-1}(f^{(n-1)}(z) - \rho w)\big| \geq \big|\rho^{-1}|f^{(n-1)}(z)| - |w|\big| > \alpha_n\,.$$

Due to Eq. (4.119), we get the following inequality for all $z \in U_n$,

$$\big|\big(f^{(n)}(z) - w\big) - \rho^{-1}\big(f^{(n-1)}(z) - \rho w\big)\big| \leq \alpha_n\,.$$

Now using Theorem 4.2.48 it follows that there exists exactly one $z \in {f^{(n-1)}}^{-1}(D^\circ_-)$ such that

$$f^{(n)}(z) - w = 0\,,$$

and we conclude that the map $f^{(n)} : {f^{(n)}}^{-1}(D^\circ_+) \to D^\circ_+$ is bijective. This completes the proof of part (a).

(b): We already showed that $U_{k+1} \subset U_k$ for all $k \in \mathbb{N}_0$. Additionally we showed that the map $f^{(k)}$ is conformal from U_{k+1} onto $D_{\rho/2}$ and that it has a unique zero z_k in the set U_k. Hence we get that U_{k+1} contains z_k and all subsequent terms of the sequence $(z_n)_{n\in\mathbb{N}}$. Therefore it suffices to show that the diameter of the sets U_n tends to zero as n tends to infinity to prove that the sequence $(z_n)_{n\in\mathbb{N}}$ converges. In the following we denote the inverse of the function $f^{(k)} : U_{k+1} \to D_{\rho/2}$ by $f^{-(k)}$. With this we can write the diameter of the set U_{n+1} as follows

$$\begin{aligned} \operatorname{diam}\big(U_{n+1}\big) &= \operatorname{diam}\big(f^{-(n)}(D_{\rho/2})\big) \\ &= \operatorname{diam}\big(f^{(\mathrm{at})} \circ f^{-(0)} \circ f^{(0)} \circ \ldots \circ f^{-(n-1)} \circ f^{(n-1)} \circ f^{-(n)}(D_{\rho/2})\big)\,, \end{aligned} \tag{4.124}$$

where we used that $z \mapsto f^{(\mathrm{at})} = E_{\mathrm{at}} - z$ is an isometry. Next we show that there exists an upper bound for the diameter in Eq. (4.124). Let $k \geq 1$, then for all $z \in D_{\rho/2}$ we have the estimate

$$|\rho z - f^{(k-1)}(f^{-(k)}(z))| \leq \rho\,\alpha_k \tag{4.125}$$

because of Eq. (4.119), and hence

$$|f^{(k-1)} \circ f^{-(k)}(z)| \le \rho\alpha_k + \frac{\rho^2}{2} \le \frac{\rho}{2} - \epsilon\,\rho\,,$$

where $\epsilon := 1/2 - \rho/2 - \alpha_1$ is positive in accordance with Eq. (4.122). Hence $f^{(k-1)} \circ f^{-(k)}$ is a map from $D_{\rho/2}$ to $D_{\rho/2-\rho\epsilon}$. Moreover, since $f^{(k)}$ and $f^{-(l)}$ are complex-valued, analytic functions for all k, $l \in \mathbb{N}$, we get by Cauchy's integral formula [86, page 3], and Eq. (4.125) that for all $z \in D_{\rho/2-\rho\epsilon}$ the following estimate holds

$$\big|\partial_z\big(f^{(k-1)} \circ f^{-(k)}(z) - \rho z\big)\big| \le \Big|\frac{1}{2\pi i}\oint_{\partial D_{\rho/2}} \frac{f^{(k-1)} \circ f^{-(k)}(w) - \rho w}{(z-w)^2}\,dw\Big| \le \frac{\alpha_k}{2\,\epsilon^2}\,. \tag{4.126}$$

It follows that $\big|\big(f^{(k-1)} \circ f^{-(k)}\big)'(z)\big| \le \rho + \alpha_k/(2\epsilon^2)$ for $z \in D_{\rho/2-\rho\epsilon}$. With a similar estimate we also get for $z \in D_{\rho/2-\rho\epsilon}$ that $\big|\big(f^{(\mathrm{at})} \circ f^{-(0)}\big)'(z)\big| \le 1 + \alpha_0/(2\rho\epsilon^2)$. Inserting these bounds into Eq. (4.124) yields

$$\begin{aligned}
\operatorname{diam}\big(U_{n+1}\big) &\le \big(1 + \frac{\alpha_0}{2\rho\epsilon^2}\big)\operatorname{diam}\big(f^{(0)} \circ f^{-(1)} \circ \ldots \circ f^{-(n-1)}(D_{\rho/2-\rho\epsilon})\big)\\
&\le \rho^{n-1}\prod_{k=0}^{n-1}\big(1 + \frac{\alpha_k}{2\rho\epsilon^2}\big)\operatorname{diam}\big(D_{\rho/2-\rho\epsilon}\big)\\
&\le \rho^n \exp\Big(\sum_{k=0}^{\infty}\frac{\alpha_k}{2\rho\epsilon^2}\Big)\,,
\end{aligned}$$

where we used for the last inequality that $1 + x \le \exp(x)$ for all $x \in \mathbb{R}$.

(c): We split the proof in three parts. First we show that

i) $U_{n+1} \cap \mathbb{R}$ is an interval for all $n \ge 0$,

ii) $\partial_x f^{(n)} < 0$ on $U_{n+1} \cap \mathbb{R}$,

and then, using i) and ii), we prove the actual statement of (c)

iii) there exists an $a < z_\infty$ such that $H^{[0]}[x]$ has a bounded inverse for $x \in (a, z_\infty)$.

Let $E_{\mathrm{at}} \in \mathbb{R}$ and $H^{[0]}[z]^* = H^{[0]}[\overline{z}]$ for all $z \in \overline{B(E_{\mathrm{at}}, \rho)}$.
i) By an induction argument we conclude that $H^{[n]}[z]^* = H^{[n]}[\overline{z}]$ for $z \in U_n$. This follows directly from Eq. (4.115) and the definition of the renormalization map $\mathcal{R}_\rho$. Hence we follow from the definition and the properties of the map $E^{(n)}$ (cf. part (a)), more precisely the fact that it is a d-dimensional diagonal matrix with entries $f^{(n)} : U_n \to \mathbb{C}$ (cf. Lemma 4.2.19), that

$$\overline{f^{(n)}(z)} = f^{(n)}(\overline{z})\,, \qquad \text{for all } z \in U_n\,.$$

Additionally we already showed in part (a) that $f^{(n)} : U_{n+1} \to D_{\rho/2}$ is a homeomorphism, and thus

$$[a_{n+1}, b_{n+1}] := (f^{(n)})^{-1}([-\rho/2, \rho/2]) = U_{n+1} \cap \mathbb{R}\,,$$

is indeed an interval which satisfies

$$E_{\mathrm{at}} - \rho < a_1 < a_2 < \ldots \le z_\infty\,.$$

ii) We again use an induction argument to prove

$$\partial_x f^{(n)}(x) < 0 \ \text{ on } [a_{n+1}, b_{n+1}]\,, \qquad \text{for all } n \in \mathbb{N}_0\,. \tag{4.127}$$

Hence we start with $n = 0$. From the assumptions of the lemma we get that $\big|f^{(0)}(z) - (E_{\mathrm{at}} - z)\big| \le \alpha_0$ for $z \in U_0$ and that $z = E_{\mathrm{at}} - \rho \in U_0$. Hence we deduce

$$\big|f^{(0)}(E_{\mathrm{at}} - \rho) - \rho\big| \le \alpha_0 < \frac{\rho}{2}\,.$$

This proves $f^{(0)}(E_{\mathrm{at}} - \rho) > \rho/2$. Now using i) and Eq. (4.117) we get that $|f^{(0)}(x)| \ge \rho/2$, for all $x \in [E_{\mathrm{at}} - \rho, a_1]$. Hence the function $f^{(0)}$ must be positive on the interval $[E_{\mathrm{at}} - \rho, a_1]$. On the other hand we know from part (a) that $f^{(0)}$ is a diffeomorphism from $[a_1, b_1]$ onto $[-\rho/2, \rho/2]$. Therefore we get that $\partial_x f^{(0)}(x) < 0$ for $x \in [a_1, b_1]$. For our induction we now assume that we already know that

$$\partial_x f^{(n-1)}(x) < 0 \quad \text{on } [a_n, b_n]. \tag{4.128}$$

Note that we again denote by $f^{-(n)}$ the inverse of the function $f^{(n)} : U_{n+1} \to D_{\rho/2}$. We set $z = 0$ in Eq. (4.126) and obtain

$$\Big|\partial_x\big(f^{(k-1)} \circ f^{-(k)}(x) - \rho x\big)\big|_{x=0}\Big| \leq \frac{\rho}{2}\frac{\rho\alpha_n}{(\rho/2)^2} \leq 2\alpha_1 < \rho\,.$$

Hence we get

$$0 < (f^{(n-1)} \circ f^{-(n)})'(0) = (\partial_x f^{(n-1)})(f^{-(n)}(0))\frac{1}{(\partial_x f^{(n)})(f^{-(n)}(0))}\,.$$

We see that $(\partial_x f^{(n)})(f^{-(n)}(0))$ has the same sign as $(\partial_x f^{(n-1)})(f^{-(n)}(0))$. Moreover this sign must be negative due to the Induction-Hypothesis (4.128). Since the map $f^{(n)} : [a_{n+1}, b_{n+1}] \to [-\rho/2, \rho/2]$ is also a diffeomorphism by part (a), we are finished.

iii) For arbitrary $n \in \mathbb{N}$ we prove that the operator $H^{[n]}[x]$ is bounded invertible for $x \in [a_n, a_{n+1})$. Due to the Feshbach property (Theorem 3.2.4 i)), it then follows that $H^{[n-1]}[x], \ldots, H^{[0]}[x]$ are also bounded invertible for $x \in [a_n, a_{n+1})$.

In part i) we showed that $U_n \cap \mathbb{R}$ is an interval for any $n \geq 1$. We set $[a_n, b_n] := U_n \cap \mathbb{R}$ and additionally deduce from i) that $a_1 < a_2 < \cdots < z_\infty$ and $\lim_{n\to\infty} a_n = z_\infty$. Now recall the decomposition (3.12) and choose $x \in [a_n, a_{n+1})$. Then the operators $H^{[n]}[x]$ and $H^{[n]}_{0,0}[x]$ are self-adjoint because by the proof of i) we have $H^{[n]}[z]^* = H^{[n]}[\overline{z}]$ for $z \in U_n$. Moreover, due to Eq. (3.13) and Eq. (4.119) we obtain

$$\big\|H^{[n]}[x]\big\| = \big\|H^{[n]}_{0,0}[x] + \big(H^{[n]}[x] - H^{[n]}_{0,0}[x]\big)\big\| \geq \big|f^{(n)}(x)\big| - \xi\gamma_n\,, \tag{4.129}$$

where we used that $\big\|H^{[n]}_{0,0}[x]\big\| \geq \big\|E^{(n)}(x)\big\| = \big|f^{(n)}(x)\big|$, since $\beta_n < 1$.

In part ii) we proved that $f^{(n)}$ is decreasing on the interval $[a_{n+1}, b_{n+1}]$ and by part (a) it has a zero in this interval. Therefore we conclude

$$f^{(n)}(a_{n+1}) > 0\,.$$

By Eq. (4.117) we obtain that $|f^{(n)}| \geq \rho/2$ on $[a_n, a_{n+1})$ and thus Eq. (4.129) implies

$$\big\|H^{[n]}[x]\big\| \geq \big(\frac{\rho}{2} - \xi\,\gamma_n\big) > \big(\frac{\rho}{2} - \xi\,\frac{\rho}{8}\big) > 0\,,$$

which proves that $H^{[n]}[x]$ is bounded invertible. This completes the proof of part c). □

Construction of the eigenvector

At the end of this subsection we show that zero is an eigenvalue of the operator $H^{[0]}[z_\infty]$. More precisely we construct an eigenvector $\varphi^{(0)}$ such that $H^{[0]}[z_\infty]\varphi^{(0)} = 0$. The same construction was done in [66] and we refer there for the proof of the following theorem.

In order to state the theorem, i.e. construct the eigenvector, we define the following auxiliary operator for $z \in U_n$, (cf. Eq. (3.5))

$$Q_n[z] := \chi_\rho - \overline{\chi}_\rho\Big(H^{[n]}_{0,0}[z] + \overline{\chi}_\rho\, W^{[n]}[z]\,\overline{\chi}_\rho\Big)^{-1}\overline{\chi}_\rho\, W^{[n]}[z]\,\chi_\rho\,,$$

where $W^{[n]} := H^{[n]} - H^{[n]}_{0,0}$. Using Eq. (4.115) and Lemma 3.2.6 we conclude that for $n \geq 1$ the following equality holds

$$H^{[n-1]}[z]\,Q_{n-1}[z]\,\Gamma^*_\rho = \big(\rho\,\Gamma^*_\rho\,\chi_1\big)H^{[n]}[z]\,.$$

Additionally, due to Theorem 3.2.4, we obtain for any $\varphi \neq 0$ with $H^{[n]}[z]\,\varphi = 0$ that $Q_{n-1}[z]\,\Gamma^*_\rho\,\varphi \neq 0$. Hence the operator $Q_{n-1}[z]\,\Gamma^*_\rho$ maps eigenvectors of $H^{[n]}[z]$ that belong to the eigenvalue 0 to eigenvectors of $H^{[n-1]}[z]$ that also belong to the eigenvalue 0.

Theorem 4.2.51 (Theorem 22, [66]). *Suppose the assumptions of Lemma 4.2.50 hold. Then the limit*

$$\varphi^{(0)} = \lim_{n\to\infty} Q_0[z_\infty]\,\Gamma_\rho^*\,Q_1[z_\infty]\,\cdots\,\Gamma_\rho^*\,Q_n[z_\infty]\,\Omega$$

exists, $\varphi^{[0]} \neq 0$ *and* $H^{[0]}[z_\infty]\,\varphi^{(0)} = 0$*. Moreover,*

$$\Big\|\,\varphi^{(0)} - Q_0[z_\infty]\,\Gamma_\rho^*\,Q_1[z_\infty]\,\cdots\,\Gamma_\rho^*\,Q_n[z_\infty]\,\Omega\,\Big\| \leq C \sum_{l=n+1}^{\infty} \gamma_l\,,$$

where $C = C(\rho,\xi,\gamma_0)$*.*

4.2.6 Proof of main result for the symmetry-protected Spin-Boson model

Before we begin with the actual proof of Theorem 4.2.26 we summarize the relevant results from the previous subsections.
Assuming that Hypotheses I, II and III hold, we proved that $(H_g(s)-z, H_0(s)-z)$ is a Feshbach pair for $\boldsymbol{\chi}(s)$ (Theorem 4.2.29) and that there exists an isospectral operator $\hat{H}_g^{(1,1)}(s,z)$, which is analytic in $\mathcal{U}$ (Theorem 4.2.34). Moreover the operator $\hat{H}_g^{(1,1)}(s,z) - \langle \hat{H}_g^{(1,1)}(s,z)\rangle_\Omega$ is an element of a specific neighborhood of the free field Hamiltonian H_f (Theorem 4.2.39). Also the renormalization transformation $\mathcal{R}_\rho$ maps elements of such a neighborhood into a related neighborhood of $P_{\mathrm{red}}H_f P_{\mathrm{red}}$ (Theorem 4.2.38, Lemma 4.2.46). Furthermore we saw that the iterated application of the renormalization transformation to the operator $\hat{H}_g^{(1,1)}(s,z)$ preserves analyticity and specific symmetry properties (Proposition 4.2.41, Proposition 4.2.42). For all $n\in\mathbb{N}$ the resulting operators

$$H_g^{[n]}[s,z] := \mathcal{R}_\rho^n\big(\hat{H}_g^{(1,1)}(s,z)\big)$$

are well-defined for all z in non-empty and shrinking sets $U_n(s)$ (Remark 4.2.45, Lemma 4.2.46). These sets converge for $n\to\infty$ to a limiting point z_∞ (Lemma 4.2.50). In addition the operator $H_g^{[0]}[s,z_\infty]$ has a non-trivial kernel (Theorem 4.2.51).
Combining these results we now prove Theorem 4.2.26. More precisely, we apply the operator-theoretic renormalization group method (Subsection 3.2.2) and use the analyticity and symmetry properties of our model to show that $H_g(s)$ has an eigenvalue and a corresponding eigenvector that both depend analytically on the parameter s.

Proof of Theorem 4.2.26. Due to Hypothesis II we have to consider two distinct cases.

1. The ground-state eigenvalue $E_{\mathrm{at}}(s)$ of the operator $H_{\mathrm{at}}(s)$ is non-degenerate for $s\in X_0$.
2. The ground-state eigenvalue $E_{\mathrm{at}}(s)$ is degenerate for $s\in X_0$ and the degeneracy is induced by a set of symmetries $\mathcal{S}$ of the Hamiltonian $H_g(s)$.

The first case coincides with the main theorem (Theorem 1) of [66] and we refer there for a detailed proof. Moreover, we note that the subsequent proof of the second case is based on that proof and contains the non-degenerate case since $d_0 = 1$ is permissible.
Proof of the 2$^{\text{nd}}$ *case:* Fix $\mu>0$ and $s_0\in\mathbb{R}^\nu\cap X$ such that Hypotheses I, II and III are satisfied. Set the constants C_χ, C_β and C_γ as in Eqns. (4.112) and (4.113). Choose $\rho\in(0,4/5)$ and an open neighborhood $X_0\subset X$ of s_0, such that $\overline{X_0}=X_0$ and sufficiently small such that $C_\gamma\rho^\mu<1$ and

$$\overline{B(E_{\mathrm{at}}(s),\rho)} \subset \{z\in\mathbb{C} : (s,z)\in\mathcal{U}\}, \qquad \text{for } s\in X_0\,. \tag{4.130}$$

We note that this is possible since the map $s\mapsto E_{\mathrm{at}}(s)$ is continuous due to Hypothesis II. Next we define $\xi := \sqrt{q}/(4C_\chi)$ and choose positive constants α_0, β_0 and γ_0 that satisfy

$$\alpha_0 < \frac{\rho}{2}\,, \quad \beta_0 \leq \frac{\rho}{8C_\chi}\,, \quad \gamma_0 \leq \frac{\rho}{8C_\chi}\,, \tag{4.131}$$

and

$$\beta_0 + \frac{C_\beta\,\rho^{-1}}{1-(C_\chi\rho^\mu)^2}\,\gamma_0^2 \leq \frac{\rho}{8C_\chi}\,. \tag{4.132}$$

Then, using Theorem 4.2.29 and 4.2.34, there exists a $g_0\in\mathbb{R}_+$ such that for all $g\in[0,g_0)$ the map $s\mapsto \hat{H}_g^{(1,1)}(s,z)$ is reflection symmetric, commutes with the set of symmetries $\mathcal{S}$ and is analytic on $\mathcal{U}$.

Moreover we know from Theorem 4.2.39 that there exists a positive constant $g_1 \in \mathbb{R}$ such that for all $g \in [0, g_1)$ and $(s,z) \in \mathcal{U}$ we have a Feshbach pair $(H_g(s) - z, H_0(s) - z)$ for $\boldsymbol{\chi}(s)$ and

$$\hat{H}_g^{(1,1)}(s,z) - (E_{\mathrm{at}}(s) - z) \in \mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0),$$

where d_0 is defined in Eq. (4.114). Now let $g \in \big[0, \min\{g_0, g_1\}\big)$ and set $U_0(s) := \{z \in \mathbb{C} : (s,z) \in \mathcal{U}\}$, then $H_g^{[0]}[s,z] := H_g^{(1,1)}(s,z)$ fulfills Assumption (A). In addition it satisfies the assumptions of Lemma 4.2.19. Next we define neighborhoods

$$\mathcal{U}_n := \big\{(s,z) \in \mathcal{U}_{n-1} : \big\|E_g^{(n-1)}(s,z)\big\| \leq \rho/8\big\},$$

with $E_g^{(n)}(s,z)$ defined as in Eq. (4.116) and we define sets $U_n(s) := \big\{z \in \mathbb{C} : (s,z) \in \mathcal{U}_n\big\}$ for $n \in \mathbb{N}$. Since the assumptions of Lemma 4.2.46 are satisfied, we conclude that the iteratively defined operators $H_g^{[n]}[s,z] := \mathcal{R}_\rho^n(H_g^{(1,1)}(s,z))$ are well-defined for $(s,z) \in \mathcal{U}_n$ and $U_n(s) \neq \emptyset$. Moreover, $H_g^{[n]}[s,z]$ is reflection symmetric, commutes with the set of symmetries $\mathcal{S}$ and is analytic on $\mathcal{U}_n^\circ(s)$ for every $n \in \mathbb{N}$. For more details see also Lemma 4.2.36, Theorem 4.2.38, and Propositions 4.2.41 and 4.2.42. Furthermore, the analytic map $E_g^{(n)}(s,z)$ is a d_0-dimensional diagonal matrix with entries $f_g^{(n)}(s,z) : U_n(s) \to \mathbb{C}$, due to Lemma 4.2.19 and the symmetry preservation property of $\mathcal{R}_\rho$. By Lemma 4.2.50, it has a unique zero $z_{n,g}(s)$ in $U_n(s)$ and for $n \to \infty$ the sequence of zeros $(z_{n,g}(s))_{n\in\mathbb{N}}$ converges to a point $z_{\infty,g}(s)$.
Now we are ready to verify the assertions of Theorem 4.2.26. In particular we show that the limit point $z_{\infty,g}(s)$ is analytic for $s \in X_0$ and that there exists an eigenvector $\psi_g(s)$ of the operator $H_g(s)$ corresponding to the eigenvalue $z_{\infty,g}(s)$ that is analytic in X_0, as well. In the third and last step we prove that $z_{\infty,g}(s_0)$ is ground-state eigenvalue of $H_g(s_0)$ for $s_0 \in X_0 \cap \mathbb{R}^\nu$.
Step 1: $z_{\infty,g}(s) = \lim_{n\to\infty} z_{n,g}(s)$ is analytic on X_0.
Since $E_g^{(n)}(s,z)$ is analytic on $U_n^\circ(s)$ and bijective in a neighborhood of $z_{n,g}(s)$ we obtain from the implicit function theorem [117, page 366] that $z_{n,g}(s)$ is analytic in $U_n^\circ(s)$, too. Moreover, for $n \to \infty$ it converges uniformly to $z_{\infty,g}(s)$ for all $s \in X_0$. Hence we deduce the analyticity of $z_{\infty,g}(s)$ on X_0 from the Weierstrass approximation theorem of complex analysis [117, page 102].
Step 2: For $s \in X_0$ there exists an eigenvector $\psi_g(s)$ of $H_g(s)$ with eigenvalue $z_{\infty,g}(s)$, such that $\psi_g(s)$ depends analytically on s.
We use the analyticity of $H_g^{[n]}[s,z]$ on $\mathcal{U}_n^\circ$ and deduce from Proposition 4.2.41, that the auxiliary operator

$$Q_{n,g}[s,z] := \chi_\rho - \overline{\chi}_\rho\Big(H_{0,0}^{[n]}[s,z] + \overline{\chi}_\rho\, W_g^{[n]}[s,z]\,\overline{\chi}_\rho\Big)^{-1}\overline{\chi}_\rho\, W_g^{[n]}[s,z]\,\chi_\rho,$$

is analytic on $\mathcal{U}_n^\circ$, as well. Note that $W_g^{[n]} := H_g^{[n]} - H_{0,0}^{[n]}$. Therefore the map $s \mapsto Q_{n,g}[s, z_{\infty,g}(s)]$ is analytic on X_0 for all $n \in \mathbb{N}$ by Step 1. Let $\varphi_{\mathrm{at}}(s_0)$ be any unit vector in the range of $P_{\mathrm{at}}(s_0)\mathcal{H}_{\mathrm{at}}$. We conclude that

$$\varphi_{n,g}^{(0)}(s) := Q_{0,g}[s, z_{\infty,g}(s)]\,\Gamma_\rho^*\, Q_{1,g}[s, z_{\infty,g}(s)]\ \cdots\ \Gamma_\rho^*\, Q_{n,g}[s, z_{\infty,g}(s)]\,(\varphi_{\mathrm{at}}(s_0) \otimes \Omega)$$

is also analytic on X_0. These vectors converge in the limit $n \to \infty$ uniformly to a vector $\varphi_g^{(0)}(s) \neq 0$ due to Theorem 4.2.51. In addition we get $H_g^{[0]}[s, z_{\infty,g}(s)]\varphi_g^{(0)}(s) = 0$ and hence $\varphi_g^{(0)}$ is analytic on X_0. By Theorem 4.2.34 (b), which is also known as the Feshbach property, the vector

$$\psi_g(s) = Q_{\boldsymbol{\chi}(s)}[s, z_{\infty,g}(s)]\, U(s)\, \varphi_g^{(0)}(s)$$

is an eigenvector of $H_g(s)$ with eigenvalue $z_{\infty,g}(s)$. We conclude that ψ_g is analytic on X_0 as well.
Step 3: For $s_0 \in X_0 \cap \mathbb{R}^\nu$ we have $z_{\infty,g}(s_0) = \inf \sigma\big(H_g(s_0)\big)$.
We already know that $H_g(s_0)$ is self-adjoint for $s_0 \in X_0 \cap \mathbb{R}^\nu$ and $\sigma\big(H_g(s_0)\big) = \big[E_g(s_0), \infty\big)$. Since $z_{\infty,g}(s_0)$ is an eigenvalue of $H_g(s_0)$ we have

$$E_g(s_0) \leq z_\infty(s_0).$$

On the other hand, since $E_{\mathrm{at}}(s_0) \in \mathbb{R}$, we have by Lemma 4.2.50 (c) that there exits a number $a < z_\infty(s_0)$ such that $H_g^{[0]}[s_0, x]$ has a bounded inverse for all $x \in (a, z_\infty(s_0))$. By Theorem 3.2.4 it follows that

$$(a, z_\infty(s_0)) \cap \sigma(H_g(s_0)) = \emptyset.$$

Therefore $z_{\infty,g}(s_0) = E_g(s_0)$ and we have finished the proof of Theorem 4.2.26. □

Remark 4.2.52. In this section we considered a quantum mechanical system consisting of atomic particles linearly coupled to a bosonic radiation field. The system was subject to specific symmetry restrictions. In particular it exhibited a degeneracy that was directly induced by a set of symmetries. Nevertheless we showed that such systems have a unique ground-state eigenvalue and that this eigenvalue depends analytically on the parameters of the system, particularly on the coupling constant g. Such analyticity results are of great importance in quantum chemistry. For example if we want to analyse the behavior of a chemical compound consisting of atomic nuclei and electrons. Such a system is way too complicated to be solved in full generality. However it can be solved approximately by the so-called *Born-Oppenheimer approximation*. For the system described above the validity of this approximation depends highly on the regularity of the ground-state energy of the system of electrons with respect to the positions of their corresponding nuclei [29, 37]. The reason for this is that the atomic nuclei move much slower than the electrons since they are much more massive. Therefore on the time-scale of nuclei-motion the electrons relax very rapidly to their corresponding ground-state configuration. Hence one first solves the equation of motion for the electrons alone where the positions of the nuclei are included as an external potential. The resulting, possibly degenerate, ground state for the electron configuration is then used to solve the equation of motion for the whole system. Without analyticity, i.e. exact knowledge of ground state and ground-state eigenvalue of the electron configuration, this approximation would not be very accurate.

Remark 4.2.53. It is worth mentioning that a result similar to Theorem 4.2.26 for the standard model of non-relativistic quantum electrodynamics can be obtained as well. The required estimates are harder to prove since for example one has to consider additional terms in the proof of the Feshbach pair. Moreover one has more complicated expressions in the construction of the operator-valued integral kernels, in particular, if one wants to include the anisotropic case as in [123]. However, in the end an application of Lemma 4.2.19 or more precisely an application of some variant of the Lemma of Schur (cf. Lemma 4.2.9) at the right moment is the crucial ingredient in the proof for the degenerate case likewise. This will be address in a forthcoming joint work with David Hasler.

4.3 The hydrogen atom in dipole approximation

In this section we give an explicit example for a quantum mechanical system that has a degenerate ground state due to a symmetry. We consider a system consisting of a static nucleus, one electron with spin and the photon field, i.e. we model a hydrogen atom in Born-Oppenheimer approximation. In the following we want to analyse the behavior of the ground state and ground-state eigenvalue of this system if an interaction between the electron and the photon field is turned on. Without loss of generality we assume that the nucleus is fixed at the origin $0 \in \mathbb{R}^3$ and that the electron mass is equal to one. Moreover we assume that the electron stays bounded to the nucleus even when it is allowed to interact with the photon field. The electron interacts with the photon field by absorbing and emitting photons. Thus high-energy photons need to be excluded from the interaction process. Due to the spin of the electron the considered system has a time reversal symmetry [104]. Therefore the ground state eigenvalue of the total energy of the system is degenerate by Kramers' degeneracy theorem (Lemma 4.2.12). Using the theory developed in Section 4.2 we will nevertheless show that the ground states and the ground-state eigenvalue are real-analytic functions of the coupling constant, see Theorem 4.3.5 below.
We consider the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F} ,$$

where $\mathcal{H}_{\mathrm{el}} := L^2(\mathbb{R}^3; \mathbb{C}^2)$ and $\mathcal{F}$ is the usual Fock space with single particle space $\mathfrak{h} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. The total energy of the static hydrogen atom is given by a Pauli-Fierz Hamiltonian. In order to define this Hamiltonian we first define the so-called *quantized vector potential*

$$A(x) = A_\Lambda(x) := \frac{1}{2\pi} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\kappa_\Lambda(k)}{\sqrt{|k|}} \epsilon_\lambda(k) \left(e^{ik\cdot x} a_\lambda(k) + e^{-ik\cdot x} a_\lambda^*(k) \right) dk ,$$

where $x \in \mathbb{R}^3$, $\epsilon_\lambda(k)$ is a real-valued, measurable polarization vector satisfying for $\lambda, \mu \in \{1, 2\}$,

$$\epsilon_\lambda(k) \cdot \epsilon_\mu(k) = \delta_{\lambda\mu} \quad \text{and} \quad k \cdot \epsilon_\lambda(k) = 0 . \tag{4.133}$$

Moreover the so-called *ultraviolet cutoff* $\kappa_\Lambda(k)$ is a smooth cutoff function that vanishes outside a ball with radius Λ. The operator-valued distributions $a_\lambda(k)$ and $a_\lambda^*(k)$ obey canonical commutation relations and we refer to Chapter 2 for more details. Furthermore we denote by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the vector of the three Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.134}$$

which act on $\mathbb{C}^2$. We choose units such that $\hbar$, c and four times the Rydberg energy are equal to one. Moreover we change the scaling by $(x, k) \mapsto (x/\alpha, \alpha^2 k)$ and express all positions as multiples of the fine-structure constant $\alpha = e^2$, where $-e$ is the electric charge of the electron. In these units the Hamiltonian of the static hydrogen atom reads

$$H_\alpha := \Big(p_{\mathrm{el}} + \alpha^{3/2} A(\alpha x)\Big)^2 + \alpha^{3/2}\, \boldsymbol{\sigma} \cdot B(\alpha x) + V(x) + H_f \tag{4.135}$$

with $p_{\mathrm{el}} := -i\,\nabla_x$, the so-called *Coulomb potential* $V(x) := -\frac{1}{|x|}$ and

$$B(x) = B_\Lambda(x) := \frac{i}{2\pi} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \kappa_\Lambda(k) \sqrt{|k|}\, \big(\frac{k}{|k|} \wedge \epsilon_\lambda(k)\big) \Big(e^{ik\cdot x} a_\lambda(k) - e^{-ik\cdot x} a_\lambda^*(k)\Big)\, dk \tag{4.136}$$

is the *quantized magnetic field.* Note that $B(x) = \operatorname{curl} A(x)$ and $\alpha^{3/2}$ now plays the role of the coupling parameter g. Similar representations of the Hamiltonian H_α are widely used in the theory of non-relativistic quantum electrodynamics, see for example [21, 56, 75, 100, 105, 123].
Unfortunately, in this generality, the Hamiltonian H_α does not fit into the framework of Spin-Boson models. Therefore we use an approximation of it which is a Spin-Boson Hamiltonian.
More precisely we use the so-called *dipole approximation.* Such an approximation does in general not yield an reasonable and meaningful result. The reason why we nonetheless get a useful result is directly related to the considered model and the spectral problem we are concerned with. Namely we are only interested in properties of the ground-state and ground-state energy. Moreover the Coulomb potential as well as the UV-cutoff in the quantized vector potential inhibit large excursions of the electron. Therefore we can assume that the charge distribution is concentrated around the origin $0 \in \mathbb{R}^3$ and replace the quantized vector potential $A(x)$ in Eq. (4.135) by the localized vector potential $A(0)$. Unfortunately a constant vector potential results in a non-existing magnetic potential and the interaction with the electron spin drops out. We therefore artificially include the magnetic potential again by thinking of it as an externally induced potential at $0 \in \mathbb{R}^3$. In a second step we apply a so-called *Pauli-Fierz transformation* to gauge away the localized vector potential $A(0)$. To be precise we unitarily transform the Hamiltonian H_α by the operator-valued transformation

$$U := \exp(-i\alpha^{3/2} A(0) \cdot x)\,,$$

applying a *commutator expansion* [7]

$$e^{-itX}\, Y\, e^{itX} = \sum_{m=0}^{\infty} \frac{(-it)^m}{m!}\, [X, Y]_m\,, \tag{4.137}$$

where $[X, Y]_m = [X, [X, Y]_{m-1}]$ and $[X, Y]_0 = Y$ for suitable self-adjoint operators X and Y.
We obtain the following transformation equations

$$\begin{aligned} U x\, U^* &= x\,, \qquad U A(0)\, U^* = A(0)\,, \\ U p_{\mathrm{el}}\, U^* &= p_{\mathrm{el}} - \alpha^{3/2} A(0)\,, \\ U a_\lambda(k)\, U^* &= a_\lambda(k) - \frac{i\alpha^{3/2}}{2\pi} \frac{\kappa_\Lambda(k)}{\sqrt{|k|}} \epsilon_\lambda(k) \cdot x\,. \end{aligned}$$

Moreover we get

$$U H_f\, U^* = H_f + \alpha^{3/2} E_\perp(0) \cdot x + \frac{\alpha^3}{\pi^2} \int_{\mathbb{R}^3} |\kappa_\Lambda(k)|^2\, dk \cdot x^2\,,$$

where $E_\perp(x) := -i[H_f, A(x)]$ is the so-called *quantized electric field.* Using Eqns. (4.133) and (4.136) we directly see that $[A(0) \cdot x, B(0)]$ is equal to zero, i.e.

$$U B(0)\, U^* = B(0)\,.$$

Inserting the transformed operators into Eq. (4.135) we get the *dipole approximation*

$$H_\alpha^{\text{approx}} := UH_\alpha U^* = H_{\text{el}} + H_f + \alpha^{3/2}E_\perp(0)\cdot x + \alpha^{3/2}\boldsymbol{\sigma}\cdot B(0) + \frac{\alpha^3}{\pi^2}\int_{\mathbb{R}^3}|\kappa_\Lambda(k)|^2\,dk\cdot x^2\,, \tag{4.138}$$

where $H_{\text{el}} := p_{\text{el}}^2 + V(x)$ denotes the unperturbed Hamiltonian of the electron-nucleus system. We refer to [19, 64] for more details on this approximation.
The *unperturbed Hamiltonian* H_{el} is bounded from below. This follows as a consequence of Sobolev's inequality since the Coulomb potential $V(x)$ is in $L^{3-\epsilon}(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ for all $\epsilon>0$ [98, 101]. In particular we have the following lower bound for $\psi\in D(H_{\text{el}})\subseteq\mathcal{H}_{\text{el}}$,

$$\|(H_{\text{el}})^{1/2}\psi\|^2 \geq -C\|\psi\|^2\,,\quad C>0\,. \tag{4.139}$$

Moreover, since we assume that the electron is confined to a small neighborhood around its nucleus we can, with out loss of generality, multiply the operator of position by a symmetric spatial-cutoff $g\in C_0^\infty(\mathbb{R}^3)$ which is zero outside a sufficiently large open ball. Thus the dipole Hamiltonian (4.138) is equivalent to

$$\widetilde{H}_\alpha^{\text{approx}} := H_{\text{el}} + H_f + \alpha^{3/2}E_\perp(0)\cdot xg(x) + \alpha^{3/2}\boldsymbol{\sigma}\cdot B(0) + \frac{\alpha^3}{\pi^2}\int_{\mathbb{R}^3}|\kappa_\Lambda(k)|^2\,dk\cdot(xg(x))^2\,.$$

The last term in the equation above is bounded and amounts to a mere shift of the spectrum of $\widetilde{H}_\alpha^{\text{approx}}$ by a real number. We therefore drop it and consider the following Hamiltonian

$$H_\alpha^{\text{dip}} := H_{\text{el}} + H_f + \alpha^{3/2}E_\perp(0)\cdot xg(x) + \alpha^{3/2}\boldsymbol{\sigma}\cdot B(0)\,. \tag{4.140}$$

This Hamiltonian is linear in creation and annihilation operator and therefore in the class of Spin-Boson models we defined in Chapter 2.
In the following we define a specific time reversal symmetry in Fock representation, cf. [104, 105]. In order to do this let $x\in\mathbb{R}^3$, $(k_j,\lambda_j)\in\mathbb{R}^3\times\{1,2\}$ for $j\in\mathbb{N}$ and $\mathfrak{s}\in\{\uparrow,\downarrow\}$. For $\varphi:=\varphi_\uparrow\oplus\varphi_\downarrow\in L^2(\mathbb{R}^3;\mathcal{F})\oplus L^2(\mathbb{R}^3;\mathcal{F})$ we define the following operator

$$J_F\varphi := j\varphi_\uparrow\oplus j\varphi_\downarrow\,,$$

where j is an involution on $L^2(\mathbb{R}^3;\mathcal{F})\oplus L^2(\mathbb{R}^3;\mathcal{F})\cong L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathcal{F}$. More precisely the operator j acts on an element $\varphi_\mathfrak{s} := \varphi_\mathfrak{s}^{(0)}(x)\oplus\bigoplus_{n=1}^\infty\varphi_\mathfrak{s}^{(n)}(x;k_1,\lambda_1,\ldots,k_n,\lambda_n)$ of $L^2(\mathbb{R}^3;\mathcal{F})$ by

$$j\varphi_\mathfrak{s} := \overline{\varphi_\mathfrak{s}^{(0)}}(-x)\oplus\bigoplus_{n=1}^\infty\overline{\varphi_\mathfrak{s}^{(n)}}(-x;k_1,\lambda_1,\ldots,k_n,\lambda_n)\,.$$

An easy computation shows that the annihilation and creation operators are *reality preserving* with respect to the operator J_F, i.e.

$$J_F\,a_\lambda(k) = a_\lambda(k)\,J_F\,,\qquad J_F\,a_\lambda^*(k) = a_\lambda^*(k)\,J_F\,.$$

As a direct consequence we get

$$\begin{aligned} J_F\,(-i\nabla_x) &= (-i\nabla_x)\,J_F\,, & J_F\,g(x)x &= -g(-x)x\,J_F\,,\\ J_F\,E_\perp(0) &= -E_\perp(0)\,J_F\,, & J_F\,B(0) &= -B(0)\,J_F\,,\\ J_F\,H_f &= H_f\,J_F\,, & J_F\,V(x) &= V(-x)\,J_F\,. \end{aligned}$$

We define a specific *time reversal symmetry* in Fock representation by

$$\vartheta_F := \sigma_2 J_F\,. \tag{4.141}$$

Proposition 4.3.1. *The time reversal symmetry ϑ_F is a symmetry for H_α^{dip}.*

Proof. According to Definition 4.2.10 we have to prove

$$\vartheta_F H_\alpha^{\text{dip}} = H_\alpha^{\text{dip}}\vartheta_F.$$

Since $V(x)=\frac{1}{|x|}=V(-x)$ this follows from an elementary matrix multiplication. □

Remark 4.3.2. It is also possible to define a time reversal symmetry in Schroedinger representation [105].

Lemma 4.3.3. *If H_α^{dip} has an eigenvector it must be at least twice degenerate.*

Proof. An easy calculation shows that $\vartheta_F^2 = -\mathbb{1}_{\mathbb{C}^2}$. Hence Kramers' degeneracy theorem (Lemma 4.2.12) proves the lemma. $\square$

From the result in [67] we know that H_α^{dip} has a ground state ψ_α for all values of α. We denote the corresponding ground-state energy by E_α.

Lemma 4.3.4. *On $\mathcal{F}$ the time reversal symmetry ϑ_F leaves the Fock vacuum invariant. Moreover, if the eigenspace, $\mathcal{V}$, of the unperturbed ground state of H_α^{dip} has dimension two, then the set of symmetries $\{\mathbb{1}, \vartheta_F\}$ acts irreducibly on $\mathcal{V}$.*

Proof. We have by definition that the operator J_F leaves the Fock vacuum invariant. Hence restricted to $\mathcal{F}$ the time reversal symmetry ϑ_F leaves the Fock vacuum invariant as well. As in the proof of Kramers degeneracy theorem (Lemma 4.2.12) we see that ϑ_F leaves $\mathcal{V}$ invariant. Now suppose that $\mathcal{W} \subset \mathcal{V}$ is an invariant subspace. If $\mathcal{W}$ is nontrivial it must contain a nonzero vector $\varphi \in \mathcal{W}$. Hence again as in the proof of Kramers' degeneracy theorem, we see that $\vartheta_F \varphi$ is nonzero and orthogonal to φ. We conclude that $\mathcal{W}$ has dimension two and hence $\mathcal{W} = \mathcal{V}$. Thus the symmetries $\{\mathbb{1}, \vartheta_F\}$ act irreducibly on $\mathcal{V}$. $\square$

The subsequent theorem proves the assertion from the beginning of this section.

Theorem 4.3.5. *Suppose $E_{\mathrm{el}} := \inf \sigma(H_{\mathrm{el}})$ is an isolated eigenvalue of H_{el}. In a neighborhood of $\alpha = 0$ the ground-state energy E_α and the corresponding eigenvectors of H_α^{dip} are real-analytic functions of $\alpha^{3/2}$.*

Proof. In order to prove the theorem we verify Hypothesis I, II and III from Section 4.2 and apply Theorem 4.2.26. Let X be the open ball with radius one around the origin in $\mathbb{R}^3$. Then for $s \in X$ and $(k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$ we define the coupling function

$$G_s(k,\lambda) := -s\,\frac{i}{2\pi}\,|k|\,\kappa_\Lambda(k)\Big(\epsilon_\lambda(k)\cdot xg(x) + \boldsymbol{\sigma}\cdot\big(\frac{k}{|k|}\wedge\epsilon_\lambda(k)\big)\Big)\,.$$

The function $G : X \to \mathcal{L}(\mathcal{H}_{\mathrm{at}}; L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{H}_{\mathrm{at}}))\,;\ s \mapsto G_s$ is uniformly bounded for $s \in X$ because $|s| < 1$, $xg(x)\psi$ is bounded for $\psi \in \mathcal{H}_{\mathrm{at}}$ and $G_s(k)$ is square-integrable for $k \in \mathbb{R}^3 \times \mathbb{Z}_2$. Moreover the mapping $s \mapsto G_s$ is analytic in s. Thus, using Lemma 4.2.21, the function G is analytic. For $0 < \mu < \frac{1}{2}$ an easy estimate proves that

$$\sup_{s\in X} \|G\|_\mu < \infty\,.$$

Therefore Hypothesis I is satisfied. Due to Proposition 4.3.1 and Lemma 4.3.4 we obtain that Hypothesis II is clearly satisfied for $H_{\mathrm{at}}(s) := H_{\mathrm{el}}$ and $s_0 := 0 \in X$. In order to use Theorem 4.2.26 it remains to verify Hypothesis III. Since H_{el} is self-adjoint and bounded from below (cf. Eq. (4.139)), we have

$$\mathrm{Re}\langle\psi, H_{\mathrm{at}}(s)\psi\rangle = \|(H_{\mathrm{el}})^{1/2}\psi\|^2 \geq -C\langle\psi,\psi\rangle,$$

for $\psi \in D(H_{\mathrm{el}})$. Hence Corollary 2 in [66], or more precisely its proof, shows that Hypothesis III holds. Therefore the conclusions of Theorem 4.2.26 hold. This concludes the proof. $\square$

5 Asymptotic expansions

In Chapter 4 we studied analytic expansions in the coupling constant of two specific Spin-Boson models. Particularly, in Section 4.2 we showed that the ground state and the ground-state energy of a specific quantum mechanical system are analytic functions of the coupling constant. Similar analyticity results have been shown in many other situations as well, see for example [1, 2, 66, 74] and the references given in the previous chapters. However in general such analyticity results, many of them obtained using operator-theoretic renormalization, are rather surprising. This comes from the fact that various quantities in quantum mechanics are calculated using perturbation theory. As long as the perturbation is 'small' compared to the unperturbed system one can expect to obtain a good approximation of the actual physical quantities (cf. Born-Oppenheimer approximation [29]). In case of isolated eigenvalues even regular perturbation theory is applicable and one can calculate eigenvalues and eigenvectors in terms of convergent power series. These series are called Rayleigh-Schrödinger perturbation series [93, 115]. However, in general, the calculation of the Rayleigh-Schrödinger expansion coefficients involve sums of divergent expressions. Hence at first sight it is not obvious in which situations these infinities eventually cancel each other and give a convergent expansion. Indeed in the situation considered in Section 4.1 we were only able to show that an analytic expansion of the ground state and ground-state eigenvalue exists in a cone with apex at the origin. Moreover there are other situations where the ground-state eigenvalue is not an analytic function of the coupling constant [25].

In this chapter we consider series expansions in the coupling constant that do not need to converge but rather yield an approximation of the ground state and ground-state eigenvalue if the series is truncated after a finite number of terms and the coupling constant tends towards zero. Such expansions are called *asymptotic expansions*. For generalized Spin-Boson models asymptotic expansions of the first few orders were investigated in [24–26,69]. More recently, Arai studied asymptotic expansion formulas up to arbitrary order in [12]. He assumed a strong infrared regularization, i.e., the higher the order of expansion the stronger the infrared regularization. In [31] we relaxed this infrared assumption significantly.

More precisely we showed that for a large class of generalized Spin-Boson models there exist asymptotic expansions for the ground state and ground-state energy up to arbitrary order requiring only a very reasonable infrared assumption. The key idea in the proof is to show that the infinities involved in calculating the Rayleigh-Schrödinger expansion coefficients cancel out.

This chapter is based on the work in [31] and is organized as follows. In Section 5.1 we introduce the model, state existence and finiteness results for the expansion coefficients of the ground state and ground-state energy and formulate the main result (Theorem 5.1.6). In Section 5.2 we derive formulas for expansion coefficients of the ground state and ground-state energy in terms of the coupling constant. Moreover we determine general conditions for which these expansion coefficients give an asymptotic expansion for a general class of models. Finally we prove the main result in Section 5.3.

Remark 5.0.1. The actual proofs of the existence/finiteness of the expansion coefficients for the ground state and the ground-state energy are rather technical and we do not present them in this thesis. All the details are given in Section 4 and 5 of [31].

5.1 Model and statement of results

In this section we introduce the model and state the results of this chapter. Let $\mathcal{H}_{\mathrm{at}}$ be a separable Hilbert space and let H_{at} be a self-adjoint operator in $\mathcal{H}_{\mathrm{at}}$. We assume that $E_{\mathrm{at}} := \inf \sigma(H_{\mathrm{at}})$ is a nondegenerate eigenvalue of H_{at}, which is isolated from the rest of the spectrum, i.e.,

$$E_{\mathrm{at}} < \epsilon_1 := \inf \left(\sigma(H_{\mathrm{at}}) \setminus \{E_{\mathrm{at}}\} \right).$$

Moreover we denote with φ_{at} the corresponding normalized eigenvector and with P_{at} the orthogonal eigenprojection of E_{at}. Let $\mathfrak{h} := L^2((\mathbb{R}^3)^n;\mathbb{C})$ and as usual we denote by $\mathcal{F}_{\mathfrak{h}}$ the symmetric Fock space. Moreover we denote by $\Omega = (1,0,0,\ldots) \in \mathcal{F}_{\mathfrak{h}}$ the vacuum vector and define the free field Hamiltonian by

$$H_f : D(H_f) \subset \mathcal{F}_{\mathfrak{h}} \to \mathcal{F}_{\mathfrak{h}}\,;\ (H_f\,\psi)_n(k_1,\ldots,k_n) := (|k_1| + |k_2| + \cdots + |k_n|)\psi_n(k_1,\ldots,k_n)\,,$$

where $D(H_f)$ denotes the domain of H_f. For more details we refer to Section 2.2.
The total Hilbert space is defined by

$$\mathcal{H} := \mathcal{H}_{\mathrm{at}} \otimes \mathcal{F}_{\mathfrak{h}} \simeq \bigoplus_{n=0}^{\infty} L^2_s((\mathbb{R}^3)^n;\mathcal{H}_{\mathrm{at}})\,, \tag{5.1}$$

where

$$L^2_s((\mathbb{R}^3)^n;\mathfrak{b}) \\ := \left\{\psi \in L^2((\mathbb{R}^3)^n;\mathfrak{b}) : \psi(k_1,\ldots,k_n) = \psi(k_{\pi(1)},\ldots,k_{\pi(n)})\ \forall \text{ permutations } \pi \text{ of } \{1,\ldots,n\}\right\},$$

for a separable Hilbert space $\mathfrak{b}$. We shall identify the spaces on the right hand side of Eq. (5.1) and occasionally drop the tensor sign in the notation. Recall that we defined in Section 3.3 for a strongly measurable function $G : \mathbb{R}^3 \to \mathcal{L}(\mathcal{H}_{\mathrm{at}})$ with

$$\int \|G(k)\|^2 dk < \infty\,,$$

the annihilation operator

$$a(G) : D(a(G)) \subset \mathcal{H} \to \mathcal{H}\,;\ \psi \mapsto (a(G)\,\psi)_n(k_1,\ldots,k_n) := \sqrt{n+1}\int G^*(k)\psi_{n+1}(k,k_1,\ldots,k_n)\,dk\,,$$

which is a densely defined closed operator. We denote its adjoint again by $a^*(G) := [a(G)]^*$, and introduce the following field operator

$$\phi(G) := \overline{a(G) + a^*(G)}\,,$$

where the line denotes the closure.
In order to define the total Hamiltonian we assume in addition that

$$\int \|G(k)\|^2(1 + |k|^{-1})\,dk < \infty\,, \tag{5.2}$$

since then it is well known that $\phi(G)$ is infinitesimally small with respect to $\mathbb{1}_{\mathcal{H}_{\mathrm{at}}} \otimes H_f$. This allows us to define the total Hamiltonian of the interacting system

$$H(\lambda) = H_{\mathrm{at}} \otimes \mathbb{1}_{\mathcal{F}_{\mathfrak{h}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{at}}} \otimes H_f + \lambda V\,, \tag{5.3}$$

as a semibounded self-adjoint operator on the domain $D(H(0))$, where $\lambda \in \mathbb{R}$ is the coupling constant and $V = \phi(G)$. We set

$$E(\lambda) = \inf \sigma(H(\lambda))\,.$$

Below we shall make the following assumption

Hypothesis (A). *There exists a positive constant λ_0 such that for all $\lambda \in [0,\lambda_0]$ the number $E(\lambda)$ is a simple eigenvalue of $H(\lambda)$ with eigenvector $\psi(\lambda) \in \mathcal{H}$.*

Remark 5.1.1. There are numerous existence results for ground states, cf. [13, 21, 53, 60, 67, 127]. In particular, it was shown that Hypothesis (A) holds if H_{at} has compact resolvent and the coupling function satisfies

$$\int \|G(k)\|^2(1 + |k|^{-2})\,dk < \infty\,. \tag{5.4}$$

A proof of this assertion is given in [60].

The following theorem shows the finiteness of the expansion coefficients for the ground-state energy. To derive the expansion coefficients one formally expands the eigenvalue equation for the ground state in powers of the coupling constant λ and inductively solve for the expansion coefficients of the ground-state energy. This results in the recursion relation (5.9), below. In Section 5.2 we explicitly deduce this formula. In order to formulate the theorem we introduce the following notations. We write

$$H_0 = H(0)\,,$$

and

$$\psi_0 = \varphi_{\mathrm{at}} \otimes \Omega\,.$$

Moreover we denote by P_0 the projection onto ψ_0 and set $\bar{P}_0 := 1 - P_0$. Let P_Ω denote the orthogonal projection in $\mathcal{F}_\mathfrak{h}$ onto Ω. Then we can write

$$\bar{P}_0 = P_{\mathrm{at}} \otimes \bar{P}_\Omega + \bar{P}_{\mathrm{at}} \otimes 1_{\mathcal{F}_\mathfrak{h}}\,, \tag{5.5}$$

where $\bar{P}_\Omega = 1_{\mathcal{F}_\mathfrak{h}} - P_\Omega$ and $\bar{P}_{\mathrm{at}} = 1_{\mathcal{H}_{\mathrm{at}}} - P_{\mathrm{at}}$.

Theorem 5.1.2. *Suppose that* (5.4) *holds. Then there exists a unique sequence* $(E_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that*

$$E_0 = E_{\mathrm{at}} \tag{5.6}$$
$$E_1 = \langle \psi_0, V\psi_0\rangle \tag{5.7}$$
$$E_n = \lim_{\eta\downarrow 0} E_n(\eta)\,, \quad n \geq 2\,, \tag{5.8}$$

where

$$E_n(\eta) := \sum_{k=2}^{n} \sum_{\substack{j_1+\cdots+j_k=n\\ j_s\geq 1}} \langle \psi_0, (\delta_{1j_1}V - E_{j_1}) \prod_{s=2}^{k} \left\{(E_0 - \eta - H_0)^{-1}\bar{P}_0(\delta_{1j_s}V - E_{j_s})\right\}\psi_0\rangle\,. \tag{5.9}$$

In particular the limit on the right hand side of (5.9) *exists and is a finite number. The sequence* $(E_n)_{n\in\mathbb{N}}$ *can be defined inductively using* (5.6)–(5.8).

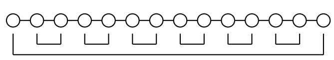
A detailed proof of this theorem is given in [31]. We provide only the idea of the proof here.

Idea of the proof. In order to prove the finiteness of the expansion coefficients (5.9) one first expresses the expansion coefficients as a sum of so-called *linked contractions* involving so-called *renormalized propagators.* These renormalized propagators take into account cancellations and do therefore have an improved infrared behavior. We comment on this in the subsequent remark. Finally one proves the finiteness of each expansion coefficient by estimating each one of these renormalized propagators.

Remark 5.1.3. First note that the positive number η that appears in Eq. (5.9) serves as a regularization and that Theorem 5.1.2 claims that the limit exists as the regularization is removed. This is not obvious since some of the individual terms on the right hand side of Eq. (5.9) diverge. In order to illustrate this let $n = 2m$ and consider the summand where $j_s = 1$ for all s. We insert Eq. (5.5) and replace V by $a^*(G) + a(G)$. Then, using Wicks theorem (Theorem 3.3.14) and the Pull-Through Formula (Lemma 3.3.11), we multiply out the resulting expression. We obtain various terms with one of them being

$$\begin{aligned}(-1)^{n-1}\int dk_1\cdots dk_m \langle\varphi_{\mathrm{at}}, G^*(k_1)\frac{P_{\mathrm{at}}}{|k_1|+\eta}\boxed{G^*(k_2)\frac{P_{\mathrm{at}}}{|k_1|+|k_2|+\eta}G(k_2)}\frac{P_{\mathrm{at}}}{|k_1|+\eta}\\ \cdots G^*(k_m)\frac{P_{\mathrm{at}}}{|k_1|+|k_m|+\eta}G(k_m)\frac{P_{\mathrm{at}}}{|k_1|+\eta}G(k_1)\varphi_{\mathrm{at}}\rangle\,,\end{aligned} \tag{5.10}$$

which is obtained by contracting the first and the last entry of the interaction and contracting the remaining nearest neighbor pairs. We symbolically pictured this by



Now, if $\eta \downarrow 0$, the integral over k_1 may become divergent for large m. An example for such a situation is the case that $\int dk|k|^{-m}\|P_{\mathrm{at}}G(k)P_{\mathrm{at}}\|^2$ diverges for m sufficiently large. The convergence of Eq. (5.9) can be restored by using cancellations originating from the energy subtractions present in the same formula. To illustrate this, consider the summand where $j_1 = 1$, $j_2 = 2$ and $j_3 = \cdots = j_{n-1} = 1$. As before we obtain various terms with one of them being the same as Eq. (5.10) except for the expression in the box which is replaced by $E_2 P_{\mathrm{at}}$. Thus adding these two terms we can factor out

$$\begin{aligned}\int dk_2 P_{\mathrm{at}} G^*(k_2) \frac{1}{|k_1|+|k_2|+\eta} G(k_2) P_{\mathrm{at}} + E_2 P_{\mathrm{at}} \\ = \int dk_2 P_{\mathrm{at}} G^*(k_2) \left(\frac{1}{|k_1|+|k_2|+\eta} - \frac{1}{|k_2|} \right) G(k_2) P_{\mathrm{at}} \\ = -(|k_1|+\eta) \int dk_2 P_{\mathrm{at}} G^*(k_2) \frac{1}{(|k_1|+|k_2|+\eta)|k_2|} G(k_2) P_{\mathrm{at}} \,, \end{aligned} \tag{5.11}$$

where we used again Eq. (5.9) to calculate E_2. One sees that replacing the expression in the box in Eq. (5.10) by the term (5.11) remedies the singularity $k_1 \to 0$. Hence in order to establish the finiteness of Eq. (5.9) one has to show that similar cancellations can be carried out at every order. More details and corresponding proofs are given in Section 4 of [31].

Theorem 5.1.4. *Suppose the assumptions of Theorem 5.1.2 hold and let $(E_n)_{n\in\mathbb{N}}$ be the unique sequence given in Theorem 5.1.2. Let*

$$\psi_0 = \varphi_{\mathrm{at}} \otimes \Omega \,.$$

Then for all $m \in \mathbb{N}$ the following limit exists

$$\psi_m = \lim_{\eta \downarrow 0} \psi_m(\eta) \,,$$

where

$$\psi_m(\eta) := \sum_{k=1}^{m} \sum_{\substack{j_1+\cdots+j_k=m \\ j_s \geq 1}} \prod_{s=1}^{k} \left\{ (E_0 - H_0 - \eta)^{-1} \bar{P}_0 (\delta_{1 j_s} V - E_{j_s}) \right\} \psi_0 \,.$$

Remark 5.1.5. This theorem shows that the expansion coefficients for the ground state exist. Its proof is analogous to that of Theorem 5.1.2, with the difference that one has to account for the square of the resolvent which may now appear in operator products. We refer to Section 5 in [31] for a detailed proof.

We saw that Theorem 5.1.2 establishes the finiteness of the expansion coefficients of the ground-state energy. Hence next we show that the expansion coefficients yield an asymptotic expansion of the ground-state energy. This is the content of the following theorem.

Theorem 5.1.6. *Suppose Condition* (5.4) *and Hypothesis (A) hold. Then the sequence $(E_n)_{n\in\mathbb{N}}$ defined in Theorem 5.1.2 yields an asymptotic expansion of the ground-state energy, i.e.,*

$$\lim_{\lambda \downarrow 0} \lambda^{-n} \left(E(\lambda) - \sum_{k=0}^{n} E_k \lambda^k \right) = 0 \,.$$

Remark 5.1.7. We want to note that if we would assume the infrared condition

$$\int \|G(k)\|^2 (1 + |k|^{-2-\mu}) dk < \infty \,,$$

for some $\mu > 0$, which is slightly stronger than Condition (5.4), then it would follow from [66] that there exists an analytic expansion of the ground-state energy. In addition, there are couplings with (5.4) for which additional symmetries may cancel infrared divergences such that the ground-state energy is analytic, cf. [72, 75].

Remark 5.1.8. In view of Remark 5.1.1 the Hypothesis (A) is not a restrictive assumption. In many situations its validity follows once Condition (5.4) is satisfied.

Remark 5.1.9. We also want to remark on the usefulness of an asymptotic expansions. While the existence of an asymptotic expansion is weaker than the existence of an analytic expansion, our result holds in situations where analytic expansions have not yet been shown. Moreover we even expect that our method can be used to derive asymptotic expansions in situations where analytic expansions in fact do not exist. Nevertheless one may gain certain spectral informations in these situations using for example Borel summability methods to recover the ground state and ground-state eigenvalue from their asymptotic expansions.

5.2 Asymptotic perturbation theory

In the following we derive formulas for the expansion coefficients of the ground state and its energy. Moreover we show that the ground-state energy has an asymptotic expansion up to some order, say n, provided the expansion coefficients for the ground state and its energy are finite up to the order n and a continuity assumption for the ground state holds. We derive this result with two different methods. The first method employs formal expansions and the comparison of coefficients combined with an analytic estimate. The second method is based on a Feshbach type argument together with a resolvent expansion. Note that we use the same symbols as before, although we state the results for more general operators than we introduced previously. Let V and H_0 be self-adjoint operators in a Hilbert space $\mathcal{H}$. In order to prove our results we use the following assumption.

Hypothesis (B). *The operator H_0 is bounded from below and V is H_0-bounded. There exists a positive constant λ_0 such that for all $\lambda \in [0, \lambda_0]$ there exists a simple eigenvalue $E(\lambda)$ of*

$$H(\lambda) = H_0 + \lambda V$$

with eigenvector $\psi(\lambda)$. Moreover,

$$\lim_{\lambda \to 0} \psi(\lambda) = \psi(0) \neq 0 \,, \qquad \lim_{\lambda \to 0} E(\lambda) = E(0)$$

and

$$\langle \psi(0), \psi(\lambda) \rangle = 1 \,, \tag{N}$$

for all $\lambda \in [0, \lambda_0]$.

We note that Condition (N) can always be achieved using a suitable normalization, possibly making the positive number λ_0 smaller. For notational convenience we shall write

$$E_0 = E(0) \,, \qquad \psi_0 = \psi(0) \,.$$

Moreover let P_0 denote the projection onto the kernel of $H_0 - E_0$ and let $\bar{P}_0 = 1 - P_0$.

5.2.1 Expansion method

The idea behind the expansion method is to expand the eigenvalue equation in a formal power series and equating coefficients. This leads to Eq. (5.12). In Lemma 5.2.1 we show that if there exists a solution of Eq. (5.12) up to some order n, then the ground-state energy has an asymptotic expansion up to the same order, provided Hypothesis (B) holds. In Lemma 5.2.2 we inductively solve Eq. (5.12), and by Lemma 5.2.3 we obtain an explicit formula for the inductive solution. We want to mention that a similar result was obtained in [12]. However, our assumptions are less restrictive in comparison to the assumptions there.

Lemma 5.2.1. *Suppose Hypothesis (B) holds. Let $n \in \mathbb{N}$ and suppose there exist $\psi_1, \ldots, \psi_n \in \bar{P}_0 \mathcal{H}$ and numbers $E_1, \ldots, E_n \in \mathbb{C}$ such that for all $m \in \mathbb{N}$ with $m \leq n$ we have*

$$H_0 \psi_m + V \psi_{m-1} = \sum_{k=0}^{m} E_k \psi_{m-k} \,. \tag{5.12}$$

Then for all $m \in \{1, \ldots, n\}$ we have that

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \left(E(\lambda) - \sum_{k=0}^{m} E_k \lambda^k \right) = 0 \,, \tag{5.13}$$

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \langle \psi_0, V(\psi(\lambda) - \sum_{k=0}^{m} \psi_k \lambda^k) \rangle = 0 \,. \tag{5.14}$$

Before we prove the lemma first observe that Eq. (5.12) implies that for all $m \leq n$ we have

$$\langle \psi_0, V \psi_{m-1} \rangle = E_m \,.$$

Proof of Lemma 5.2.1. We prove this by induction in n. For this we define for $\lambda \in (0, \lambda_0)$ the quantities

$$e_n(\lambda) := \lambda^{-n}(E(\lambda) - (E_0 + \lambda E_1 + \lambda^2 E_2 + \cdots + \lambda^n E_n)),$$
$$\rho_n(\lambda) := \lambda^{-n}(\psi(\lambda) - (\psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots + \lambda^n\psi_n)).$$

Hence Eq. (5.14) for $m = 0$ is just Hypothesis (B). It remains to show the induction step. The eigenvalue equation gives

$$\bar{P}_0(H(\lambda) - E(\lambda))\bar{P}_0\psi(\lambda) = -\bar{P}_0 V P_0 \psi(\lambda). \tag{5.15}$$

$n-1 \to n$: Suppose that Eq. (5.12) holds for all $m \in \{1, \ldots, n\}$. By the induction Hypothesis we know that $\lambda E_n + \lambda e_n(\lambda) \to 0$ and $\langle V\psi_0, \lambda\psi_n + \lambda\rho_n(\lambda)\rangle \to 0$. From the eigenvalue equation we find

$$(H_0 + \lambda V)\left[\sum_{k=0}^{n} \lambda^k \psi_k + \lambda^n \rho_n(\lambda)\right] = \left(\sum_{k=0}^{n} \lambda^k E_k + \lambda^n e_n(\lambda)\right)\left[\sum_{k=0}^{n} \lambda^k \psi_k + \lambda^n \rho_n(\lambda)\right].$$

By ordering according to powers of λ we see from Eq. (5.12) that many terms vanish and

$$\lambda V\psi_n + (H_0 + \lambda V)\rho_n(\lambda) = \rho_n(\lambda)E(\lambda) + e_n(\lambda)\sum_{k=0}^{n} \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^{n} E_j \psi_{k-j}. \tag{5.16}$$

If one applies P_0 to Eq. (5.16) one obtains

$$\lambda P_0 V(\psi_n + \rho_n(\lambda)) = e_n(\lambda)\psi_0.$$

By induction Hypothesis the left hand side tends to zero as $\lambda \to 0$. This shows that Eq. (5.13) holds for all $m \in \{1, \ldots, n\}$. Solving for terms involving $\rho_n(\lambda)$ in Eq. (5.16) we arrive at

$$(H(\lambda) - E(\lambda))\rho_n(\lambda) = e_n(\lambda)\sum_{k=0}^{n} \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^{n} E_j \psi_{k-j} - \lambda V\psi_n.$$

Now applying $\bar{P}_0$ to this equation and using that $P_0\rho_n(\lambda) = 0$ we find

$$\bar{P}_0(H(\lambda) - E(\lambda))\bar{P}_0\rho_n(\lambda) = \bar{P}_0\left(e_n(\lambda)\sum_{k=0}^{n} \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^{n} E_j \psi_{k-j} - \lambda V\psi_n\right).$$

Calculating the inner product with $\psi(\lambda)$ and using Eq. (5.15) we find

$$\langle \psi(\lambda), P_0 V \rho_n(\lambda)\rangle \;= -\langle \bar{P}_0\psi(\lambda), e_n(\lambda)\sum_{k=1}^{n} \lambda^{k-1}\psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n-1} \sum_{j=k-n}^{n} E_j \psi_{k-j} - \bar{P}_0 V\psi_n\rangle.$$

This and Hypothesis (B) imply that Eq. (5.14) holds for all $m \in \{1, \ldots, n\}$. □

Next we inductively solve Equation (5.12).

Lemma 5.2.2. *(Inductive Formula) Let $n \in \mathbb{N}$ and suppose there exist $\psi_1, \ldots, \psi_n \in \bar{P}_0\mathcal{H}$ and numbers $E_1, \ldots, E_n \in \mathbb{C}$ such that for all $m \in \mathbb{N}$ with $m \le n$ we have*

$$H_0\psi_m + V\psi_{m-1} = \sum_{k=0}^{m} E_k \psi_{m-k}. \tag{5.17}$$

Then defining

$$E_{n+1} := \langle \psi_0, V\psi_n\rangle, \tag{5.18}$$

as well as

$$\psi_{n+1} := (H_0 - E_0)^{-1}\bar{P}_0\left(\sum_{k=1}^{n+1} E_k \psi_{n+1-k} - V\psi_n\right), \tag{5.19}$$

provided

$$\bar{P}_0\left(\sum_{k=0}^{n} E_{k+1}\psi_{n-k} - V\psi_n\right) \in \operatorname{dom}\left((H_0 - E_0)^{-1}\bar{P}_0\right), \tag{5.20}$$

we obtain a solution of Eq. (5.17) *for $m = n + 1$.*

We note that the assumption in Eq. (5.20) is less restrictive than the corresponding one in [12]. This will turn out to be crucial to obtain the asymptotic expansion of the ground state to arbitrary order.

Proof. The lemma follows by insertion of Eqns. (5.19) and (5.18) into Eq. (5.17) for $m = n + 1$. □

By solving the recursive relation of the previous lemma, one obtains the following formulas.

Lemma 5.2.3. *(Direct Formula) Let $n \in \mathbb{N}$ and suppose there exist $\psi_1, \dots, \psi_n \in \bar{P}_0\mathcal{H}$ and numbers $E_1, \dots, E_n \in \mathbb{C}$ such that the following holds. We have $E_1 = \langle \psi_0, V\psi_0\rangle$, and for all $m \in \mathbb{N}$ with $2 \le m \le n$ we have*

$$E_m = -\sum_{k=2}^{m} \sum_{\substack{j_1+\dots+j_k=m \\ j_s \ge 1}} \langle \psi_0, (E_{j_1} - \delta_{1j_1}V) \prod_{s=2}^{k} \left\{ (H_0 - E_0)^{-1}\bar{P}_0(E_{j_s} - \delta_{1j_s}V) \right\} \psi_0 \rangle , \tag{5.21}$$

moreover for all $m \in \mathbb{N}$ with $m \le n$ we have

$$\psi_m = \sum_{k=1}^{m} \sum_{\substack{j_1+\dots+j_k=m \\ j_s \ge 1}} \prod_{s=1}^{k} \left\{ (H_0 - E_0)^{-1}\bar{P}_0(E_{j_s} - \delta_{1j_s}V) \right\} \psi_0 , \tag{5.22}$$

assuming that the expressions on the right hand side of Eq. (5.21) and (5.22) exist in the sense of Lemma 5.2.2. Then for all $m \in \mathbb{N}$ with $m \le n$ we have

$$H_0\psi_m + V\psi_{m-1} = \sum_{k=0}^{m} E_k \psi_{m-k} .$$

Proof. We prove this lemma by induction in n. The case $n = 1$ follows by straight forward calculation. Now suppose the claim holds for n. Then we know in addition that the assumption of Lemma 5.2.2 holds. Hence we can define E_{n+1} as in Eq. (5.18)

$$\begin{aligned} E_{n+1} &:= \langle \psi_0, V\psi_n \rangle \\ &= -\sum_{k=2}^{n+1} \sum_{\substack{j_1+\dots+j_k=n+1 \\ j_s \ge 1}} \langle \psi_0, (E_{j_1} - \delta_{1j_1}V) \prod_{s=2}^{k} \left\{ (H_0 - E_0)^{-1}\bar{P}_0(E_{j_s} - \delta_{1j_s}V) \right\} \psi_0 \rangle , \end{aligned}$$

where we used the assumption (5.22) in the second line (note that $\langle \psi_0, E_j\bar{P}_0(\,\cdot\,)\rangle = 0$). In the same way as in Eq. (5.19) we define

$$\begin{aligned} \psi_{n+1} &:= (H_0 - E_0)^{-1}\bar{P}_0 \left(\sum_{j=1}^{n+1} (E_j - \delta_{1j}V)\psi_{n+1-j} \right) \\ &= \sum_{k=1}^{n+1} \sum_{\substack{j_1+\dots+j_k=n+1 \\ j_s \ge 1}} \prod_{s=1}^{k} \left\{ (H_0 - E_0)^{-1}\bar{P}_0(E_{j_s} - \delta_{1j_s}V) \right\} \psi_0 , \end{aligned}$$

where in the first line a slightly different notation than in Eq. (5.19) is used and in the second line we again employ the assumption (5.22). Now using Lemma 5.2.2 the claim of the lemma holds for $n + 1$ as well. □

5.2.2 Resolvent method

For the second method we use a Feshbach type or Schur complement argument (cf. Section 3.2) together with a resolvent expansion. We note that the proof of the following lemma is inspired by [12].

Lemma 5.2.4. *Suppose that Hypothesis (B) holds. Starting with $K_0 := \frac{\bar{P}_0}{H_0 - E_0}$ and $E_1 := \langle \psi_0, V\psi_0\rangle$, we assume that we can define recursively for $m \in \{1, \dots, n-2\}$*

$$\begin{aligned} K_m &:= \sum_{j=1}^{m} K_{j-1}(E_{m+1-j} - \delta_{jm}V)K_0 , \qquad (5.23) \\ E_{m+1} &:= -\langle \psi_0, VK_{m-1}V\psi_0 \rangle , \end{aligned}$$

such that $\bar{P}_0 V\psi_0 \in \mathrm{dom}(K_l)$ *for* $l = 0, \dots, n-2$. *Then* $E(\lambda)$ *has an asymptotic expansion up to order* n, *i.e., for all* $m = 1, \dots, n$

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \left(E(\lambda) - \sum_{k=0}^{m} E_k \lambda^k \right) = 0.$$

Remark 5.2.5. The statement of Lemma 5.2.4 is equivalent to the statements of Lemma 5.2.3 and Lemma 5.2.1 combined. In particular, we may solve iteratively for K_m and obtain the relation

$$VK_{m-2}V = \sum_{k=2}^{m} \sum_{\substack{j_1+\cdots+j_k=m \\ j_s \geq 1}} (E_{j_s} - \delta_{1j_s} V) \prod_{s=2}^{k} \left\{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s} V) \right\}.$$

Moreover, given $E(\lambda)$, we can recover $\psi(\lambda)$ by

$$\psi(\lambda) = \psi_0 - \lambda \bar{P}_0 \left[\bar{P}_0 (H(\lambda) - E(\lambda)) \bar{P}_0 \big|_{\mathrm{Ran}(\bar{P}_0)} \right]^{-1} \bar{P}_0 V P_0 \psi_0.$$

Proof. The eigenvalue equation $H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda)$ can be split into the equivalent system of equations

$$P_0\big(\lambda V + E_0 - E(\lambda)\big) P_0 \psi(\lambda) + \lambda P_0 V \bar{P}_0 \psi(\lambda) = 0\,, \tag{5.24a}$$
$$\lambda \bar{P}_0 V P_0 \psi(\lambda) + \bar{P}_0 \big(H(\lambda) - E(\lambda)\big) \bar{P}_0 \psi(\lambda) = 0\,, \tag{5.24b}$$

by applying the projections P_0 and $\bar{P}_0$ respectively. From Eq. (5.24a) we obtain that

$$\frac{E(\lambda) - E_0}{\lambda} \langle \psi_0, P_0 \psi(\lambda) \rangle - \langle \psi_0, V P_0 \psi(\lambda) \rangle = \langle V\psi_0, \bar{P}_0 \psi(\lambda) \rangle = o(1),$$

i.e.

$$\frac{E(\lambda) - E_0}{\lambda} \xrightarrow{\lambda \to 0} \langle \psi_0, V\psi_0 \rangle.$$

This shows the claim for $n = 1$. Now we show the lemma by induction. Suppose the claim holds for n and the assumptions of the lemma hold for $n+1$. Then the recursively defined functions

$$\begin{aligned} E^{[0]}(\lambda) &:= E(\lambda)\,, \\ E^{[k]}(\lambda) &:= \frac{E^{[k-1]}(\lambda) - E_{k-1}}{\lambda}\,, \end{aligned} \tag{5.25}$$

satisfy

$$\lim_{\lambda \downarrow 0} E^{[k]}(\lambda) = E_k\,, \qquad \text{for } k = 0, \dots, n\,.$$

We write the part $\bar{P}_0 \psi(\lambda)$ as follows

$$\begin{aligned} \bar{P}_0 \psi(\lambda) &= \frac{\bar{P}_0}{H_0 - E_0} (H_0 - E_0) \bar{P}_0 \psi(\lambda) \\ &= \frac{\bar{P}_0}{H_0 - E_0} \big[H(\lambda) - E(\lambda) + (E(\lambda) - E_0 - \lambda V) \big] \bar{P}_0 \psi(\lambda). \end{aligned}$$

Eq. (5.24b) now implies

$$\bar{P}_0 \psi(\lambda) = \lambda \frac{\bar{P}_0}{H_0 - E_0} \left[-V P_0 \psi(\lambda) + (E^{[1]}(\lambda) - V) \bar{P}_0 \psi(\lambda) \right]. \tag{5.26}$$

Iterated insertion of Eq. (5.26) into itself and terminating the expansion after we have reached order λ^n leads to the following claim.

Claim: We have for $k = 1, \dots, n$

$$P_0 V \bar{P}_0 \psi(\lambda) = P_0 V \sum_{j=1}^{k} -\lambda^j K_{j-1} V P_0 \psi(\lambda) + P_0 V \lambda^k R_k(\lambda) \bar{P}_0 \psi(\lambda)\,, \tag{5.27}$$

where $R_k(\lambda)$ is defined by

$$R_k(\lambda) := \sum_{j=1}^{k} K_{j-1}(E^{[k+1-j]}(\lambda) - \delta_{jk} V) . \tag{5.28}$$

We note that these expressions are well defined by the assumption $\bar{P}_0 V \psi_0 \in \mathrm{dom}(K_l)$. In the following we prove the claim. Equation (5.27) for $k = 1$ is just Equation (5.26) multiplied by $P_0 V$. Now assume that Eq. (5.27) is true for a specific $k \leq n - 1$. We insert first the Definition (5.28) and then Definition (5.25) to arrive at

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0 \psi(\lambda) &= P_0 V \sum_{j=1}^{k} K_{j-1}(E^{[k+1-j]}(\lambda) - \delta_{jk} V) \bar{P}_0 \psi(\lambda) \\ &= P_0 V \sum_{j=1}^{k} \Big(K_{j-1}(E_{k+1-j} - \delta_{jk} V) \bar{P}_0 \psi(\lambda) + \lambda K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0 \psi(\lambda) \Big) . \end{aligned}$$

Next we insert Eq. (5.26) in the first summand and obtain

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0 \psi(\lambda) \; = \lambda P_0 V \sum_{j=1}^{k} \Big(& K_{j-1}(E_{k+1-j} - \delta_{jk} V) K_0 \big(-V P_0 \psi(\lambda) + (E^{[1]}(\lambda) - V) \bar{P}_0 \psi(\lambda) \big) \\ & + K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0 \psi(\lambda) \Big). \end{aligned}$$

Using Eq. (5.23) we find

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0 \psi(\lambda) &= \lambda P_0 V \Big(K_k \big(-V P_0 \psi(\lambda) + (E^{[1]}(\lambda) - V) \bar{P}_0 \psi(\lambda) \big) + \sum_{j=1}^{k} K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0 \psi(\lambda) \Big) \\ &= -\lambda P_0 V K_k V P_0 \psi(\lambda) + \lambda P_0 V \sum_{j=1}^{k+1} K_{j-1} \big(E^{[k+2-j]}(\lambda) - \delta_{j,k+1} V \big) \bar{P}_0 \psi(\lambda) . \end{aligned}$$

By Eq. (5.28) this expression agrees with Eq. (5.27) if k is replaced by $k + 1$. Hence inserting this expression into Eq. (5.27) with k we obtain Eq. (5.27) with k replaced by $k + 1$. This proves the claim. Next we insert the claim for $k = n$ into Eq. (5.24a) to conclude

$$\left(P_0(E^{[1]}(\lambda) - V) P_0 + \sum_{j=1}^{n} \lambda^j P_0 V K_{j-1} V P_0 \right) P_0 \psi(\lambda) = \lambda^n P_0 V R_n(\lambda) \bar{P}_0 \psi(\lambda) . \tag{5.29}$$

Taking the inner product with ψ_0 and using the induction hypothesis (5.23), we obtain

$$\begin{aligned} E^{[1]}(\lambda) \; - \; E_1 - \sum_{j=1}^{n} \lambda^j E_{j+1} &= \lambda^n \langle \psi_0, P_0 V R_n(\lambda) \bar{P}_0 \psi(\lambda) \rangle \\ &= \lambda^n \langle R_n(\lambda) V \psi_0, \bar{P}_0 \psi(\lambda) \rangle . \end{aligned}$$

Dividing by λ^n we find using Eq. (5.25)

$$\lambda^{-(n+1)} \left(E(\lambda) - \sum_{j=0}^{n+1} \lambda^j E_j \right) = \langle R_n(\lambda) V \psi_0, \bar{P}_0 \psi(\lambda) \rangle = o(1) .$$

This shows the assertion of the lemma for $n + 1$. □

Remark 5.2.6. Note that Eq. (5.29) implies

$$\left(E(\lambda) - \sum_{j=0}^{n} \lambda^j H_j \right) P_0 \psi(\lambda) = o(\lambda^n) ,$$

for $H_1 := P_0 V P_0$ and $H_n := -P_0 V K_{n-2} V P_0$. In the case that H_j is diagonalizable we can choose the coefficients E_j in a suitable way from the eigenvalues corresponding to H_j. Hence the expansion equation above is valid for a degenerate perturbation theory as well.

5.3 Existence of an asymptotic expansion

In this section we present a proof for Theorem 5.1.6. To begin with we show that the ground state and the ground-state energy are continuous functions of the coupling constant, that is we verify Hypothesis (B). We recall the notation $\psi_0 = \varphi_{\rm at} \otimes \Omega$ and $E_0 = E_{\rm at}$.

Proposition 5.3.1. *Let $H(\lambda)$ be given as in Eq. (5.3) and assume that Hypothesis (A) is satisfied. Then the following holds.*

(a) *If Eq. (5.2) holds, then the ground-state energy $E(\lambda)$ satisfies $E(\lambda) \leq E_0$ and*

$$E(\lambda) - E_0 = O(|\lambda|^2)\,, \quad (\lambda \to 0)\,.$$

(b) *If Eq. (5.4) holds, then the operator $H(\lambda)$ has an eigenvector $\psi(\lambda)$ with eigenvalue $E(\lambda)$ such that*

$$\|\psi(\lambda) - \psi_0\| = O(|\lambda|)\,, \quad (\lambda \to 0)$$

and $\langle \psi_0, \psi(\lambda)\rangle = 1$ for λ in a neighborhood of zero.

Proof. (a). First we show the upper bound

$$E(\lambda) \leq \langle \psi_0, H(\lambda)\psi_0\rangle = \langle \psi_0, (H_f + H_{\rm at} + \lambda\phi(G))\,\psi_0\rangle = E_{\rm at} = E_0\,.$$

In order to show the lower bound we complete the square

$$\begin{aligned} H(\lambda) &= H_{\rm at} + H_f + \lambda\phi(G) \\ &= H_{\rm at} + \int |k| \left[a(k) + \lambda\frac{G(k)}{|k|}\right]^* \left[a(k) + \lambda\frac{G(k)}{|k|}\right] dk - |\lambda|^2 \int \frac{G(k)^* G(k)}{|k|}\, dk \\ &\geq E_{\rm at} - |\lambda|^2 \int \frac{\|G(k)\|^2}{|k|}\, dk\,. \end{aligned}$$

(b) This is a consequence of the following two claims. We write $\widehat{\psi}(\lambda) := \frac{\psi(\lambda)}{\|\psi(\lambda)\|}$.

Claim 1: We have that $\|\bar{P}_\Omega \widehat{\psi}(\lambda)\| = O(|\lambda|)$.

Calculating a commutator we obtain

$$H(\lambda)a(k)\psi(\lambda) = \big([H(\lambda), a(k)] + a(k)H(\lambda)\big)\psi(\lambda) = \big(-|k|a(k) - \lambda G(k) + a(k)H(\lambda)\big)\psi(\lambda)\,.$$

Solving for $a(k)\psi(\lambda)$ we find

$$\big(H(\lambda) - E(\lambda) + |k|\big)a(k)\psi(\lambda) = -\lambda G(k)\psi(\lambda)\,,$$

and by inversion we find for $k \neq 0$ that

$$a(k)\psi(\lambda) = -\lambda \frac{|k|}{H(\lambda) - E(\lambda) + |k|}\frac{G(k)}{|k|}\psi(\lambda)\,.$$

Thus we obtain for the number operator N the expectation

$$\begin{aligned} \langle \psi(\lambda), N\psi(\lambda)\rangle &= \int \|a(k)\psi(\lambda)\|^2\, dk \\ &= |\lambda|^2 \int \left\| \frac{|k|}{H(\lambda) - E(\lambda) + |k|}\frac{G(k)}{|k|}\psi(\lambda)\right\|^2 dk \\ &\leq |\lambda|^2 \int \frac{\|G(k)\|^2}{|k|^2} dk\; \|\psi(\lambda)\|^2\,. \end{aligned}$$

Inserting this into the inequality

$$\|\bar{P}_\Omega \psi\|^2 \leq \langle \psi, N\psi\rangle$$

we find that

$$\|\bar{P}_\Omega \widehat{\psi}(\lambda)\| = O(\lambda), \quad (\lambda \to 0)\,.$$

This shows Claim 1.

Claim 2: Let $\bar{P}_{\rm at} = 1 - P_{\rm at}$. Then we have $\|\bar{P}_{\rm at}\widehat{\psi}(\lambda)\| = O(|\lambda|)$.

We apply $\bar{P}_{\rm at}$ to the eigenvalue equation and obtain

$$\bar{P}_{\rm at}H(\lambda)\bar{P}_{\rm at}\psi(\lambda) + \bar{P}_{\rm at}H(\lambda)P_{\rm at}\psi(\lambda) = E(\lambda)\bar{P}_{\rm at}\psi(\lambda)\,.$$

Solving for terms involving $\bar{P}_{\rm at}\psi(\lambda)$ we find

$$\big(\bar{P}_{\rm at}H(\lambda)\bar{P}_{\rm at} - E(\lambda)\bar{P}_{\rm at}\big)\bar{P}_{\rm at}\psi(\lambda) = -\bar{P}_{\rm at}H(\lambda)P_{\rm at}\psi(\lambda)\,. \tag{5.30}$$

Below we show that we can invert the operator on the left hand side of Eq. (5.30). Then we estimate the corresponding inverse operator using a Neumann expansion. For this let $\epsilon_1 := \inf\sigma(H_{\rm at}|_{{\rm Ran}\bar{P}_{\rm at}})$. By (a) we have in the sense of operators on the range of $\bar{P}_{\rm at}$ that

$$(H(0) - E(\lambda))\bar{P}_{\rm at} \geq (H(0) - E_0)\bar{P}_{\rm at} = (H_{\rm at} + H_f - E_0)\bar{P}_{\rm at} \geq (\epsilon_1 - E_0)\bar{P}_{\rm at}\,.$$

Thus $\big(H(0) - E(\lambda)\big)\bar{P}_{\rm at}$ is invertible as an operator in ${\rm Ran}\bar{P}_{\rm at}$. From the proof of Lemma 3.3.12 we recall the standard estimates

$$\begin{aligned}\|a(G)\psi\| &\leq \left(\int \frac{\|G(k)\|^2}{|k|}dk\right)^{1/2} \|H_f^{1/2}\psi\|\,,\\ \|a^*(G)\psi\|^2 &\leq \int \|G(k)\|^2 dk\|\psi\|^2 + \int \frac{\|G(k)\|^2}{|k|}dk\,\|H_f^{1/2}\psi\|^2\,.\end{aligned}$$

These estimates imply that

$$\|(H_f+1)^{-1/2}\phi(G)\| = \|\phi(G)(H_f+1)^{-1/2}\| < \infty\,.$$

By (a) we find that

$$\begin{aligned}\big\|(\bar{P}_{\rm at}(H(0) - E(\lambda))\bar{P}_{\rm at})^{-1}\bar{P}_{\rm at}\phi(G)\big\| &\leq \big\|(\bar{P}_{\rm at}(H(0) - E(\lambda))\bar{P}_{\rm at})^{-1}(H_f+1)^{1/2}\big\|\big\|(H_f+1)^{-1/2}\phi(G)\big\|\\ &\leq \sup_{r\geq 0}\left|\frac{(r+1)^{1/2}}{r+\epsilon_1 - E_0}\right| \big\|(H_f+1)^{-1/2}\phi(G)\big\| =: C_G\,.\end{aligned} \tag{5.31}$$

By Neumann's theorem (Theorem 3.2.9) it follows from Eq. (5.31) that $\bar{P}_{\rm at}(H(\lambda) - E(\lambda))\bar{P}_{\rm at}$ is invertible on ${\rm Ran}\bar{P}_{\rm at}$, if $|\lambda| < C_G^{-1}$, and we have

$$\big(\bar{P}_{\rm at}(H(\lambda) - E(\lambda))\bar{P}_{\rm at}\big)^{-1} = \sum_{n=0}^{\infty}\big[-(\bar{P}_{\rm at}(H(0) - E(\lambda))\bar{P}_{\rm at})^{-1}\lambda\phi(G)\big]^n\,(\bar{P}_{\rm at}(H(0) - E(\lambda))\bar{P}_{\rm at})^{-1}\,.$$

Inserting this expression into Eq. (5.30) and using again Eq. (5.31) we find

$$\big\|\bar{P}_{\rm at}\widehat{\psi}(\lambda)\big\| = \big\|\big[\bar{P}_{\rm at}\big(H(\lambda) - E(\lambda)\big)\bar{P}_{\rm at}\big]^{-1}\bar{P}_{\rm at}H(\lambda)P_{\rm at}\widehat{\psi}(\lambda)\big\| \leq \frac{|\lambda|C_G}{1 - |\lambda|C_G}\big\|P_{\rm at}\widehat{\psi}(\lambda)\big\|\,.$$

This shows Claim 2.

Statement (b) now follows from Claims 1 and 2 by writing

$$\begin{aligned}\widehat{\psi}(\lambda) - \psi_0\langle\psi_0, \widehat{\psi}(\lambda)\rangle &= \widehat{\psi}(\lambda) - P_\Omega\otimes P_{\rm at}\widehat{\psi}(\lambda)\\ &= \bar{P}_\Omega\widehat{\psi}(\lambda) + P_\Omega\otimes\bar{P}_{\rm at}\widehat{\psi}(\lambda) \to 0\,,\end{aligned}$$

where the first term on the right hand side tends to zero because of Claim 1 and the second term because of Claim 2. Thus $\psi(\lambda) = \widehat{\psi}(\lambda)\langle\psi_0, \widehat{\psi}(\lambda)\rangle^{-1}$ is well defined for λ sufficiently close to zero and satisfies Statement (b). □

Now we are ready to prove the theorem.

Proof of Theorem 5.1.6. First we show using Theorem 5.1.2 and 5.1.4 that

$$H_0\psi_{n+1}(0) + V\psi_n(0) = \sum_{k=0}^{n+1} E_k\psi_{n+1-k}(0)\,. \tag{5.32}$$

From the convergence of $\psi_n(\eta)$ as $\eta \downarrow 0$ we obtain from the definition of E_n that

$$E_n = \langle V\psi_0, \psi_n(0)\rangle = \lim_{\eta\downarrow 0}\langle V\psi_0, \psi_n(\eta)\rangle \,. \tag{5.33}$$

From the definition of $\psi_n(\eta)$ (cf. Eq. (5.19)) we see that

$$(H_0 - E_0 + \eta)\psi_{n+1}(\eta) = \bar{P}_0 \left(\sum_{k=1}^{n+1} E_k \psi_{n+1-k}(\eta) - V\psi_n(\eta) \right) \,. \tag{5.34}$$

We claim that the limit $\eta \downarrow 0$ yields

$$(H_0 - E_0)\psi_{n+1}(0) = \bar{P}_0 \left(\sum_{k=1}^{n+1} E_k \psi_{n+1-k}(0) - V\psi_n(0) \right) \,. \tag{5.35}$$

This clearly holds for $n = 0$. Suppose that it holds for all $m \leq n-1$. Then for n the right hand side of Eq. (5.34) converges to the right hand side of Eq. (5.35). Since H_0 is a closed operator it follows that the left hand side of Eq. (5.34) converges to the left hand side of Eq. (5.35). Now Eqns. (5.35) and (5.33) imply Eq. (5.32). By Proposition 5.3.1 and Eq. (5.32) the assumptions of Lemma 5.2.1 are satisfied. Hence Theorem 5.1.6 now follows from Lemma 5.2.1. □

Remark 5.3.2. We mentioned in Remark 4.1.27 that one can recover from an asymptotic expansion under very specific conditions and using certain methods the original function. Moreover we remarked in this section (cf. Remark 5.2.6) that the presented method to acquire such an asymptotic expansion can be extended to degenerate situations. Furthermore in many situations our result requires only the reasonable infrared condition (5.4) to be applicable. Hence we expect that our technique can be used to gain valuable insight to spectral problems of degenerate systems that are out of reach for other methods. For example in situations like in Section 4.1 where the ground-state eigenvalue only depends analytically on the coupling constant in a cone with apex at the origin.

Bibliography

[1] Abdelmalek Abdesselam, *The ground state energy of the massless spin-boson model*, Ann. Henri Poincaré **12** (2011), no. 7, 1321–1347. MR 2846670

[2] Abdelmalek Abdesselam and David Hasler, *Analyticity of the ground state energy for massless Nelson models*, Comm. Math. Phys. **310** (2012), no. 2, 511–536. MR 2890307

[3] W. K. Abou Salem, Jérémy Faupin, Jürg Fröhlich, and Israel Michael Sigal, *On the theory of resonances in non-relativistic quantum electrodynamics and related models*, Adv. in Appl. Math. **43** (2009), no. 3, 201–230. MR 2549576

[4] J. Aguilar and J. M. Combes, *A class of analytic perturbations for one-body schrödinger hamiltonians*, Comm. Math. Phys. **22** (1971), no. 4, 269–279.

[5] Anton Amann, *Molecules coupled to their environment*, pp. 3–22, Springer US, Boston, MA, 1991.

[6] Laurent Amour and Jérémy Faupin, *Hyperfine splitting in non-relativistic QED: uniqueness of the dressed hydrogen atom ground state*, Comm. Math. Phys. **319** (2013), no. 2, 425–450. MR 3037583

[7] Werner O. Amrein, Anne Boutet de Monvel, and Vladimir Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Progress in Mathematics, vol. 135, Birkhäuser Verlag, Basel, 1996. MR 1388037

[8] Asao Arai, *On a model of a harmonic oscillator coupled to a quantized, massless, scalar field. I, II*, J. Math. Phys. **22** (1981), no. 11, 2539–2548, 2549–2552. MR 640665

[9] ———, *Rigorous theory of spectra and radiation for a model in quantum electrodynamics*, J. Math. Phys. **24** (1983), no. 7, 1896–1910. MR 709529

[10] ———, *Spectral analysis of a quantum harmonic oscillator coupled to infinitely many scalar bosons*, J. Math. Anal. Appl. **140** (1989), no. 1, 270–288. MR 997857

[11] ———, *Perturbation of embedded eigenvalues: a general class of exactly soluble models in Fock spaces*, Hokkaido Math. J. **19** (1990), no. 1, 1–34. MR 1039462

[12] ———, *A new asymptotic perturbation theory with applications to models of massless quantum fields*, Ann. Henri Poincaré **15** (2014), no. 6, 1145–1170. MR 3205748

[13] Asao Arai and Masao Hirokawa, *On the existence and uniqueness of ground states of a generalized spin-boson model*, J. Funct. Anal. **151** (1997), no. 2, 455–503. MR 1491549

[14] Volker Bach, Thomas Chen, Jürg Fröhlich, and Israel Michael Sigal, *Smooth Feshbach map and operator-theoretic renormalization group methods*, J. Funct. Anal. **203** (2003), no. 1, 44–92. MR 1996868

[15] ———, *The renormalized electron mass in non-relativistic quantum electrodynamics*, J. Funct. Anal. **243** (2007), no. 2, 426–535. MR 2289695

[16] Volker Bach, Jürg Fröhlich, and Alessandro Pizzo, *Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field*, Comm. Math. Phys. **264** (2006), no. 1, 145–165. MR 2212219

[17] ———, *Infrared-finite algorithms in QED. II. The expansion of the groundstate of an atom interacting with the quantized radiation field*, Adv. Math. **220** (2009), no. 4, 1023–1074. MR 2483715

[18] Volker Bach, Jürg Fröhlich, and Israel Michael Sigal, *Mathematical theory of nonrelativistic matter and radiation*, Lett. Math. Phys. **34** (1995), no. 3, 183–201. MR 1345551

[19] ———, *Quantum electrodynamics of confined nonrelativistic particles*, Adv. Math. **137** (1998), no. 2, 299–395. MR 1639713

[20] ———, *Renormalization group analysis of spectral problems in quantum field theory*, Adv. Math. **137** (1998), no. 2, 205–298. MR 1639709

[21] ———, *Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field*, Comm. Math. Phys. **207** (1999), no. 2, 249–290. MR 1724854

[22] Miguel Ballesteros, Jérémy Faupin, Jürg Fröhlich, and Baptiste Schubnel, *Quantum electrodynamics of atomic resonances*, Comm. Math. Phys. **337** (2015), no. 2, 633–680. MR 3339159

[23] E. Balslev and J. M. Combes, *Spectral properties of many-body schrödinger operators with dilatation-analytic interactions*, Comm. Math. Phys. **22** (1971), no. 4, 280–294.

[24] Jean-Marie Barbaroux, Thomas Chen, Vitali Vougalter, and Semjon Vugalter, *On the ground state energy of the translation invariant Pauli-Fierz model*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 1057–1064. MR 2361881

[25] ———, *Quantitative estimates on the binding energy for hydrogen in non-relativistic QED*, Ann. Henri Poincaré **11** (2010), no. 8, 1487–1544. MR 2769703

[26] Jean-Marie Barbaroux, Thomas Chen, and Semjon Vugalter, *Binding conditions for atomic N-electron systems in non-relativistic QED*, Ann. Henri Poincaré **4** (2003), no. 6, 1101–1136. MR 2031161

[27] H. A. Bethe, *The electromagnetic shift of energy levels*, Phys. Rev. **72** (1947), 339–341.

[28] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, Authorized English edition. Revised and enlarged by the authors. Translated from the Russian by G. M. Volkoff. Interscience Monographs in Physics and Astronomy, Vol. III, Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London, 1959. MR 0110471

[29] M. Born and R. Oppenheimer, *Zur quantentheorie der molekeln*, Annalen der Physik **389** (1927), no. 20, 457–484.

[30] Anne Boutet de Monvel and Jaouad Sahbani, *On the spectral properties of the spin-boson Hamiltonians*, Lett. Math. Phys. **44** (1998), no. 1, 23–33. MR 1623754

[31] Gerhard Bräunlich, David Hasler, and Markus Lange, *On asymptotic expansions in spin boson models*, arXiv:1608.06270 [math-ph] (2016), 50.

[32] Thomas Chen, *Infrared renormalization in non-relativistic QED and scaling criticality*, J. Funct. Anal. **254** (2008), no. 10, 2555–2647. MR 2406687

[33] Thomas Chen, Jürg Fröhlich, and Alessandro Pizzo, *Infraparticle scattering states in nonrelativistic quantum electrodynamics. II. Mass shell properties*, J. Math. Phys. **50** (2009), no. 1, 012103, 34. MR 2492583

[34] ———, *Infraparticle scattering states in non-relativistic QED. I. The Bloch-Nordsieck paradigm*, Comm. Math. Phys. **294** (2010), no. 3, 761–825. MR 2585987

[35] C. Cohen-Tannoudji, B. Diu, and F. Laloe, *Quantum mechanics, volume 1*, Wiley-VCH, June 1986.

[36] ———, *Quantum mechanics, volume 2*, Wiley-VCH, June 1986.

[37] J. M. Combes, P. Duclos, and R. Seiler, *The born-oppenheimer approximation*, pp. 185–213, Springer US, Boston, MA, 1981.

[38] Wojciech De Roeck, Jürg Fröhlich, and Alessandro Pizzo, *Absence of embedded mass shells: Cerenkov radiation and quantum friction*, Ann. Henri Poincaré **11** (2010), no. 8, 1545–1589. MR 2769704

[39] Wojciech De Roeck, Marcel Griesemer, and Antti Kupiainen, *Asymptotic completeness for the massless spin-boson model*, Adv. Math. **268** (2015), 62–84. MR 3276589

[40] Wojciech De Roeck and Antti Kupiainen, *Approach to ground state and time-independent photon bound for massless spin-boson models*, Ann. Henri Poincaré **14** (2013), no. 2, 253–311. MR 3028040

[41] Jan Dereziński, *Introduction to representations of the canonical commutation and anticommutation relations*, Large Coulomb systems, Lecture Notes in Phys., vol. 695, Springer, Berlin, 2006, pp. 63–143. MR 2497821

[42] Jan Dereziński and Vojkan Jakšić, *Spectral theory of Pauli-Fierz operators*, J. Funct. Anal. **180** (2001), no. 2, 243–327. MR 1814991

[43] P. A. M. Dirac, *The principles of quantum mechanics*, Oxford: Clarendon Press, 1947.

[44] A. Einstein, *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt*, Annalen der Physik **322** (1905), 132–148.

[45] T. Estermann, *Complex numbers and functions*, London : University of London, Athlone Press, 1962 (English), Includes bibliography.

[46] Jérémy Faupin, *Resonances of the confined hydrogen atom and the Lamb-Dicke effect in non-relativistic QED*, Ann. Henri Poincaré **9** (2008), no. 4, 743–773. MR 2413202

[47] Jérémy Faupin, Jürg Fröhlich, and Baptiste Schubnel, *Analyticity of the self-energy in total momentum of an atom coupled to the quantized radiation field*, J. Funct. Anal. **267** (2014), no. 11, 4139–4196. MR 3269873

[48] Jérémy Faupin, Møller, Jacob Schach, and E. Skibsted, *Regularity of bound states*, Reviews in Mathematical Physics **23** (2011), no. 05, 453–530.

[49] ———, *Second order perturbation theory for embedded eigenvalues*, Comm. Math. Phys. **306** (2011), no. 1, 193–228. MR 2819424

[50] Jérémy Faupin and Israel Michael Sigal, *On Rayleigh scattering in non-relativistic quantum electrodynamics*, Comm. Math. Phys. **328** (2014), no. 3, 1199–1254. MR 3201223

[51] Herman Feshbach, *Unified theory of nuclear reactions*, Ann. Physics **5** (1958), 357–390. MR 0116975

[52] Jürg Fröhlich, *On the infrared problem in a model of scalar electrons and massless, scalar bosons*, Ann. Inst. H. Poincaré Sect. A (N.S.) **19** (1973), 1–103. MR 0368649

[53] ———, *Existence of dressed one electron states in a class of persistent models*, Fortschritte der Physik **22** (1974), no. 3, 159–198.

[54] Jürg Fröhlich, Marcel Griesemer, and Benjamin Schlein, *Asymptotic completeness for Compton scattering*, Comm. Math. Phys. **252** (2004), no. 1-3, 415–476. MR 2104885

[55] ———, *Rayleigh scattering at atoms with dynamical nuclei*, Comm. Math. Phys. **271** (2007), no. 2, 387–430. MR 2287910

[56] Jürg Fröhlich, Marcel Griesemer, and Israel Michael Sigal, *Spectral theory for the standard model of non-relativistic QED*, Comm. Math. Phys. **283** (2008), no. 3, 613–646. MR 2434740

[57] Jürg Fröhlich, Marcel Griesemer, and Israel Michael Sigal, *On spectral renormalization group*, Rev. Math. Phys. **21** (2009), no. 4, 511–548. MR 2528042

[58] ———, *Spectral renormalization group and local decay in the standard model of non-relativistic quantum electrodynamics*, Rev. Math. Phys. **23** (2011), no. 2, 179–209. MR 2786226

[59] V. Georgescu, C. Gérard, and J.S. Møller, *Spectral theory of massless pauli-fierz models*, Communications in Mathematical Physics **249** (2004), no. 1, 29–78.

[60] Christian Gérard, *On the existence of ground states for massless Pauli-Fierz Hamiltonians*, Ann. Henri Poincaré **1** (2000), no. 3, 443–459. MR 1777307

[61] Fritz Gesztesy, Konstantin A Makarov, and Eduard Tsekanovskii, *An addendum to krein's formula*, Journal of Mathematical Analysis and Applications **222** (1998), no. 2, 594 – 606.

[62] Joel Gilmore and Ross H McKenzie, *Spin boson models for quantum decoherence of electronic excitations of biomolecules and quantum dots in a solvent*, Journal of Physics: Condensed Matter **17** (2005), no. 10, 1735.

[63] James Glimm and Arthur Jaffe, *Quantum physics*, second ed., Springer-Verlag, New York, 1987, A functional integral point of view. MR 887102

[64] Marcel Griesemer, *Non-relativistic matter and quantized radiation*, Large Coulomb systems, Lecture Notes in Phys., vol. 695, Springer, Berlin, 2006, pp. 217–248. MR 2497823

[65] Marcel Griesemer and David Hasler, *On the smooth Feshbach-Schur map*, J. Funct. Anal. **254** (2008), no. 9, 2329–2335. MR 2409163

[66] ———, *Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation*, Ann. Henri Poincaré **10** (2009), no. 3, 577–621. MR 2519822

[67] Marcel Griesemer, Elliott H. Lieb, and Michael Loss, *Ground states in non-relativistic quantum electrodynamics*, Invent. Math. **145** (2001), no. 3, 557–595. MR 1856401

[68] V. V. Grushin, *Les problèmes aux limites dégénérés et les opérateurs pseudo-différentiels*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 737–743. MR 0509187

[69] Christian Hainzl and Robert Seiringer, *Mass renormalization and energy level shift in non-relativistic QED*, Adv. Theor. Math. Phys. **6** (2002), no. 5, 847–871 (2003). MR 1974588

[70] David Hasler and Ira Herbst, *Absence of ground states for a class of translation invariant models of non-relativistic QED*, Comm. Math. Phys. **279** (2008), no. 3, 769–787. MR 2386727

[71] ———, *On the self-adjointness and domain of Pauli-Fierz type Hamiltonians*, Rev. Math. Phys. **20** (2008), no. 7, 787–800. MR 2436496

[72] ———, *Convergent expansions in non-relativistic qed: analyticity of the ground state*, J. Funct. Anal. **261** (2011), no. 11, 3119–3154. MR 2835993

[73] ———, *Ground state properties in non-relativistic QED*, Mathematical results in quantum physics, World Sci. Publ., Hackensack, NJ, 2011, pp. 203–207. MR 2885173

[74] ———, *Ground states in the spin boson model*, Ann. Henri Poincaré **12** (2011), no. 4, 621–677. MR 2787765

[75] ———, *Smoothness and analyticity of perturbation expansions in qed*, Adv. Math. **228** (2011), no. 6, 3249–3299. MR 2844943

[76] ———, *Uniqueness of the ground state in the Feshbach renormalization analysis*, Lett. Math. Phys. **100** (2012), no. 2, 171–180. MR 2912479

[77] David Hasler, Ira Herbst, and Matthias Huber, *On the lifetime of quasi-stationary states in non-relativistic QED*, Ann. Henri Poincaré **9** (2008), no. 5, 1005–1028. MR 2438505

[78] David Hasler and Markus Lange, *Renormalization analysis for degenerate ground states*, arXiv:1704.06966 [math-ph] (2017), 45.

[79] Ira W. Herbst, *Dilation analyticity in constant electric field. I. The two body problem*, Comm. Math. Phys. **64** (1979), no. 3, 279–298. MR 520094

[80] Ira W. Herbst and B. Simon, *Dilation analyticity in constant electric field. II. N-body problem, Borel summability*, Comm. Math. Phys. **80** (1981), no. 2, 181–216. MR 623157

[81] Einar Hille and Ralph S. Phillips, *Functional analysis and semi-groups*, American Mathematical Society Colloquium Publications, vol. 31, American Mathematical Society, Providence, R. I., 1957, rev. ed. MR 0089373

[82] Masao Hirokawa, *Remarks on the ground state energy of the spin-boson model. An application of the Wigner-Weisskopf model*, Rev. Math. Phys. **13** (2001), no. 2, 221–251. MR 1818537

[83] ______, *Ground state transition for two-level system coupled with Bose field*, Phys. Lett. A **294** (2002), no. 1, 13–18. MR 1888824

[84] Fumio Hiroshima, *Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants*, Ann. Henri Poincaré **3** (2002), no. 1, 171–201. MR 1891842

[85] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory*, Applied Mathematical Sciences, vol. 113, Springer-Verlag, New York, 1996, With applications to Schrödinger operators. MR 1361167

[86] Lars Hörmander, *An introduction to complex analysis in several variables*, third ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR 1045639

[87] James S. Howland, *The Livsic matrix in perturbation theory*, J. Math. Anal. Appl. **50** (1975), 415–437. MR 0374957

[88] Matthias Hübner and Herbert Spohn, *Spectral properties of the spin-boson Hamiltonian*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), no. 3, 289–323. MR 1335060

[89] W. Hunziker, *Distortion analyticity and molecular resonance curves*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), no. 4, 339–358. MR 880742

[90] W. Hunziker and I. M. Sigal, *The quantum N-body problem*, J. Math. Phys. **41** (2000), no. 6, 3448–3510. MR 1768629

[91] Vojkan Jakšić and Claude-Alain Pillet, *On a model for quantum friction. I. Fermi's golden rule and dynamics at zero temperature*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), no. 1, 47–68. MR 1313360

[92] ______, *On a model for quantum friction. II. Fermi's golden rule and dynamics at positive temperature*, Comm. Math. Phys. **176** (1996), no. 3, 619–644. MR 1376434

[93] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452

[94] Hendrik A. Kramers, *Théorie générale de la rotation paramagnétique dans les cristaux*, Proceedings Koninklijke Akademie van Wetenschappen **33** (1930), 959–972.

[95] M. Krein, *On Hermitian operators whose deficiency indices are* 1, C. R. (Doklady) Acad. Sci. URSS (N.S.) **43** (1944), 323–326. MR 0011170

[96] N. M. Kroll and W. E. Lamb, *On the self-energy of a bound electron*, Physical Review **75** (1949), 388–398.

[97] Willis E. Lamb and Robert C. Retherford, *Fine structure of the hydrogen atom by a microwave method*, Phys. Rev. **72** (1947), 241–243.

[98] Elliott H. Lieb and Michael Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997. MR 1415616

[99] ______, *Existence of atoms and molecules in non-relativistic quantum electrodynamics*, Adv. Theor. Math. Phys. **7** (2003), no. 4, 667–710. MR 2039034

[100] ______, *A note on polarization vectors in quantum electrodynamics*, Comm. Math. Phys. **252** (2004), no. 1-3, 477–483. MR 2104886

[101] Elliott H. Lieb and Robert Seiringer, *The stability of matter in quantum mechanics*, Cambridge University Press, Cambridge, 2010. MR 2583992

[102] M. S. Livšic, *On the scattering matrix of an intermediate system*, Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 67–70. MR 0086622

[103] Michael Loss, Tadahiro Miyao, and Herbert Spohn, *Lowest energy states in nonrelativistic QED: atoms and ions in motion*, J. Funct. Anal. **243** (2007), no. 2, 353–393. MR 2289693

[104] ______, *Kramers degeneracy theorem in nonrelativistic QED*, Lett. Math. Phys. **89** (2009), no. 1, 21–31. MR 2520177

[105] ______, *Time reversal symmetries and properties of ground states in nonrelativistic QED*, Applications of renormalization group methods in mathematical sciences, RIMS Kôkyûroku Bessatsu, B21, Res. Inst. Math. Sci. (RIMS), Kyoto, 2010, pp. 35–44. MR 2792021

[106] Tadahiro Miyao and Herbert Spohn, *Spectral analysis of the semi-relativistic Pauli-Fierz Hamiltonian*, J. Funct. Anal. **256** (2009), no. 7, 2123–2156. MR 2498761

[107] Jacob S. Möller and Matthias Westrich, *Regularity of eigenstates in regular Mourre theory*, J. Funct. Anal. **260** (2011), no. 3, 852–878. MR 2737399

[108] Edward Nelson, *Analytic vectors*, Annals of Mathematics **70** (1959), no. 3, 572–615.

[109] Carl Neumann, *Theorie der Bessel'schen Functionen : ein Analogon zur Theorie der Kugelfunctionen*, 1867.

[110] M. A. Neumark, *On spectral functions of a symmetric operator*, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7** (1943), 285–296. MR 0010790

[111] Leonardo A. Pachón and Paul Brumer, *Physical basis for long-lived electronic coherence in photosynthetic light-harvesting systems*, The Journal of Physical Chemistry Letters **2** (2011), no. 21, 2728–2732.

[112] Michael E Peskin and Daniel V Schroeder, *An introduction to quantum field theory; 1995 ed.*, Westview, Boulder, CO, 1995, Includes exercises.

[113] M. Planck, *Ueber das Gesetz der Energieverteilung im Normalspectrum*, Annalen der Physik **309** (1901), 553–563.

[114] Michael Reed and Barry Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR 0493420

[115] ______, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR 0493421

[116] ______, *Methods of modern mathematical physics. III*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979, Scattering theory. MR 529429

[117] ______, *Methods of modern mathematical physics. I*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis. MR 751959

[118] Jun John Sakurai, *Modern quantum mechanics; rev. ed.*, Addison-Wesley, Reading, MA, 1994.

[119] Manfred Salmhofer, *Renormalization*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1999, An introduction. MR 1658669

[120] Wilhelm Schlag, *A course in complex analysis and Riemann surfaces*, Graduate Studies in Mathematics, vol. 154, American Mathematical Society, Providence, RI, 2014. MR 3186310

[121] Issai Schur, *Neue Begründung der Theorie der Gruppencharaktere*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin. **Jan-Juni 1905** (1905), 26.

[122] Franz Schwabl, *Quantum mechanics*, german ed., Springer, Berlin, 2007. MR 2374994

[123] Israel Michael Sigal, *Ground state and resonances in the standard model of the non-relativistic QED*, J. Stat. Phys. **134** (2009), no. 5-6, 899–939. MR 2518974

[124] Barry Simon, *Quadratic form techniques and the balslev-combes theorem*, Comm. Math. Phys. **27** (1972), no. 1, 1–9.

[125] Herbert Spohn, *Ground state(s) of the spin-boson Hamiltonian*, Comm. Math. Phys. **123** (1989), no. 2, 277–304. MR 1002040

[126] ———, *Asymptotic completeness for Rayleigh scattering*, J. Math. Phys. **38** (1997), no. 5, 2281–2296. MR 1447858

[127] ———, *Ground state of a quantum particle coupled to a scalar Bose field*, Lett. Math. Phys. **44** (1998), no. 1, 9–16. MR 1623746

[128] ———, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, 2004. MR 2097788

[129] J. Stark, *Beobachtungen über den Effekt des elektrischen Feldes auf Spektrallinien. I. Quereffekt*, Annalen der Physik **348** (1914), no. 7, 965–982.

[130] John Von Neumann, *Mathematische grundlagen der quantenmechanik*, Springer, 1932 (ger).

[131] Steven Weinberg, *The quantum theory of fields*, Cambridge University Press, 1995.

[132] H. Weyl, *Quantenmechanik und Gruppentheorie*, Zeitschrift für Physik **46** (1927), 1–46.

[133] G. C. Wick, *The evaluation of the collision matrix*, Phys. Rev. **80** (1950), 268–272.

[134] Kenneth G. Wilson, *The renormalization group: critical phenomena and the Kondo problem*, Rev. Modern Phys. **47** (1975), no. 4, 773–840. MR 0438986

[135] Ning Wu and Yang Zhao, *Dynamics of a two-level system under the simultaneous influence of a spin bath and a boson bath*, The Journal of Chemical Physics **139** (2013), no. 5, 054118.

[136] Kosaku Yosida, *Functional analysis*, 6th ed., Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 1995.

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Bei der Auswahl und Auswertung des Materials sowie der Herstellung des Manuskripts haben mich, durch ihr mitwirken an gemeinsamen Arbeiten die Teil dieser Dissertation sind, folgende Personen unterstützt:

- Prof. Dr. David Hasler
- Dr. Gerhard Bräunlich

Ich habe die gleiche, eine in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung bereits bei einer anderen Hochschule als Dissertation eingereicht: Nein

Jena, den 20. Oktober 2017

Markus Lange, Verfasser